A SECOND COURSE IN ALGEBRA

A Text and Exercise Book With Tables

BY

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"Algebra is the intellectual instrument which has been created for rendering clear the quantitative aspect of the world."

ALFRED NORTH WHITEHEAD

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2. For the Eighth School Year
3. Everyday Algebra
4. A Second Course in Algebra

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EDITOR'S INTRODUCTION

THEORY and practice, understanding and skill, appreciation and technique, are often opposed. In some human activities they are to a certain extent independent of each other. A man may be a skillful driver and yet know little or nothing of the theory of an automobile engine. In mathematics, however, the two, understanding and technique, are so intertwined that either suffers by a lack of the other. This is particularly true in the case of algebra.

The boys or girls who attempt to master the technique of algebra without understanding the meaning of the symbols they use and without appreciating the reasons for the rules they follow (and unfortunately many do make this attempt) lack the power to apply their skill in any significant way. And the boy or girl who hasn't acquired the necessary skill at a given stage of his or her development has difficulty in understanding the next step in the development of the theory. This inevitable dependence of theory on technique and of technique on theory is too often lost sight of by pupils and teachers alike. To the man who aspires to write a textbook on algebra, it presents a most difficult problem.

This brings me to the first reason why I am glad to have the present text appear in the series I am editing: *Mr. Barber*, it seems to me, *has succeeded more than any other author with whose work I am familiar in throwing this interdependence of understanding and skill into high relief.*

His means for doing so is a carefully worked-out "question-and-answer" method which is characteristic of all of his books. It is as old as Socrates, and yet it is something distinctly new in modern textbooks on algebra. To develop it into a really effective method of presentation required skill and patience of an unusual order. As in the case of Mr. Barber's Junior High

School Series, the present text has gone through several experimental editions which have been *systematically* tested in the classroom by a group of able teachers under varying conditions. These experimental editions have been successively revised in the light of this classroom experience. This has taken much time and required much painstaking labor — but the result, I believe, has amply justified the effort: the pupil using this text will be induced *to think for himself, to understand;* he will gain a mastery of the necessary technique illuminated by this understanding, and will, therefore, have acquired those mental habits on which alone real power to use algebraic methods can develop. This is in my opinion the outstanding feature of the present text.

There is another feature which strongly appeals to me. The text is unusually *correct.* All good texts should, of course, be correct. But any scholarly teacher will realize how difficult it often is to be mathematically accurate and pedagogically sensible at the same time. Algebra abounds in such apparent conflicts: the concept of negative numbers and the laws governing their use, the division by zero, the "sum" of an infinite geometric progression, etc., are topics in which many texts are, for pedagogical reasons or for reasons less flattering to their authors, inaccurate. Any one of sound scholarship who has attempted to write a text on algebra knows how difficult some of these situations are. Mr. Barber has, I believe, made a notable contribution also in this aspect of textbook writing.

J. W. Young

PREFACE

This course is designed to follow any good first course in algebra. It provides a clearly designated minimum which can be covered in one half year by pupils of fair ability, and a maximum course which provides a full year's work for any pupils.

It has been thoroughly tested by the author and other teachers, not only with able, college-preparatory pupils in the Boston Girls' Latin School, in the Phœnix (Arizona) Union High School and Junior College, in the Pawtucket (R.I.) High School, in the Phillips Exeter Academy, and elsewhere; but also with many general and commercial pupils who have elected mathematics without thought of preparation for college.

In addition to the standard excellencies which are to be expected in the best modern texts, such as clear and scholarly presentation, an abundance of well-graded exercises, and general accord with the recommendations of the National Committee, this text presents the following special features:

1. Inductive presentation and introductory questions particularly designed to arouse intellectual curiosity and to stimulate the pupil to think for himself.
2. Special opportunity for the appreciation of mathematics. The importance of algebra and of its methods in the modern scientific world is emphasized in many ways. The scientific method of thought is consciously applied by the pupil to topic after topic.
3. Thorough teaching of the necessary manipulation of symbols, but from the point of view which shows the significance of the symbols and the purposes they serve.

PART I. THE TEXT

There is a new order of chapters. This is because the course is organized around a group of questions to which the student is seeking answers. Briefly these questions are: What are the fundamental notions which give to algebra its importance?

How does algebra make its studies and state its conclusions? How are these conclusions used? The course is, then, not a series of topics, each to be studied and set aside, but a succession of fields of application of the fundamental notions and methods of algebra, each application adding something to the answers to these questions.

Chapter I, The Nature and Purpose of Algebra, gives the student command of the fundamental tools of algebra — the symbols, equations, and tables. This means not only a review — the kind of review that is not satisfied with the recollection of rules but insists on understanding and explanation — but also some extension of the student's knowledge; as, for instance, about radicals and about quadratic equations. (By learning to solve quadratics by factoring and by formula, he is enabled, prior to the study of the theory of quadratics in Chapter X, to handle all the equations which arise.) This introductory chapter may also be found effective in creating the attitude of interest and appreciation which prepares the way for successful learning.

Chapters II–IV, Numerical Trigonometry, and the *Progressions,* come early in the course because they give the student unexcelled opportunity to discover, in fields which are new to him, how algebra proceeds to study its problems and to apply its conclusions. In the trigonometry the computations suggest the need for logarithms, and *Chapter III,* which begins with a theoretical study of *exponents,* leads to the practical use of *logarithms* and concludes with their application to trigonometry. This early introduction of logarithms makes for facility in their use and for economy in the computations of the later chapters. Attention is given to those facts about approximation which should be common knowledge.

Chapter V, Problem Solution. The solution of verbal problems is made use of throughout the course as the best means of teaching the algebraic method (or more generally the scientific method) of dealing with number relations. The *Plan of Problem*

Solution (pages 169–171) will be found to increase the student's power, to leave him never at a loss for a next step to take, and also to form habits which are invaluable in all his thinking about number situations.

Chapters VI, VII, Factoring, Fractions. From the beginning of the course the student has factored and has transformed fractions. In studying Chapter VI he organizes his knowledge of factoring and gives attention to classification, generalization, and the recognition of type forms. In Chapter VII further contact is made between algebraic and arithmetic fractions and then the emphasis is placed upon proportion in preparation for Chapter VIII. (The ratio concept has been considered at length in Chapter I.) It is interesting to see that when the inert ideas concerning fractions and factoring are no longer required in the course, these two chapters can be shortened and postponed as is required by the principles of organization on which this text is based.

Chapter VIII, Dependence. There is almost complete agreement among authorities that the function concept should play a considerable part in secondary school mathematics, but much less agreement as to how this should be brought about. As a contribution in this field the text makes frequent use of the words *number relations.* In studying formulas, in investigating progressions, in solving problems, and in many other places the student's attention is specifically directed to the number relations involved. *Number relations* are listed in Chapter I as one of the fundamental notions of the course. This development of the idea, which comes to its climax in Chapter VIII, enables the student with little expenditure of time and with no feeling of being out of his depth, to acquire this very useful mathematical point of view.

All the graphs of the course are collected in this chapter and referred to when needed elsewhere.

Chapters IX–XIII. See the Table of Contents.

PART II. EXERCISES

Part I, the Text, contains the introductions, discussions, illustrative examples, other textual material, and enough practice work to prepare the student to attack the exercises of Part II with competence and in a good attitude of mind. Part II contains 61 exercises with more than 130 subdivisions, each with a specific and obvious drill purpose. These drill exercises are separated from the text and placed by themselves where they serve all the purposes of an exercise book. They encourage the student to look up references for himself. They require him to work in the absence of models and explanations and yet provide ample cross references so that he can find help when he needs it.

NOTES ON THE USE OF THE BOOK

A time schedule. Sufficient material is provided for a full year's work, but the omission of the starred material and the following of a time schedule somewhat as follows makes possible the meeting of the requirements of, for example, the College Entrance Examination Board in a minimum of 90 days. Chapter I, 10 days; Chapter II, 5 days; Chapter III, 14 days; IV, 14 days; V, 13 days; VI, 5 days; VII, 5 days; VIII, 9 days; IX, 4 days; X, 4 days; XI, 3 days; review, 4 days. Total, 90 days.

Plans of procedure. The division of the book into two parts, text and exercises, makes possible an unusual variety of teaching plans.

The author is accustomed to devote much class time to preliminary study and discussion of the introductions and investigations of Part I. The students read silently or one student reads aloud and the class finds answers to the questions asked in the text and to other related questions. Effort is made to establish a good attitude of mind toward the topic under discussion and good habits of study. Occasionally for variety the text is assigned for out-of-class work without previous preparation.

Another plan is to assign the *exercises* without introduction and thus train the student to refer back to the text whenever he finds it necessary. This plan makes for self-reliance but it may discourage students of lesser ability and it tends to make the class period less inspiring. The author uses it sparingly.

Still another plan is for the teacher to make his classroom development independent of the text. While this requires more labor on the part of the teacher, it is preferred by some for at least a part of the time. The two-part arrangement of the book is helpful to those who follow this plan.

In any case full advantage can be taken of the opportunity for variety and the opportunity to let the student study into each topic for himself and then to do the exercises without having immediately before him either models or rules.

Studies of errors. Teachers who wish to make studies of student difficulties will find helpful the following table which was used to discover points insufficiently covered in the experimental editions of the text. It shows weaknesses of the student and of the presentation; it points to specific remedies.

Name or number of the test....................

Question	1	2	3	4	......	Average
Pupil						
A						
B						
C						
Average						

Acknowledgments. The labor necessary to make this new text has been lightened by the enthusiastic participation of students and also by stimulating contacts with mathematicians and educators who have been interested in furthering such a project. The underlying ideas and methods have been approved by such scholars as Dr. E. V. Huntington of Harvard, George W. Evans of Boston, and Dr. J. W. Young of Dartmouth. Dr. Young has given much more than the usual editorial supervision. To the

following teachers, most of whom have used the experimental editions in their classes, I am indebted: Henry M. Wright, English High School, Boston; Lena M. Perrigo, Memorial High School for Girls, Boston; H. D. Gaylord, Browne and Nichols School, Cambridge, Massachusetts; Dr. P. R. Crosby, Pawtucket (Rhode Island) High School; Elizabeth J. Martin, Charlestown (Massachusetts) High School; Paul E. Elicker, Newton (Massachusetts) High School; Myra Downs, Phœnix (Arizona) High School and Junior College; A. C. Ewen, Dean Academy, Franklin, Massachusetts; William M. Gaylor, Morris High School, New York City; Joseph B. Orleans, George Washington High School, New York City; John Bechtel, South Philadelphia High School for Boys, Philadelphia, Pa.; and particularly to George T. Major, of the Phillips Exeter Academy.

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Aims of Part II: *To serve all the purposes of an exercise book. To lead the student to work without models and explanations immediately before him. To teach him with the help of cross-references and the index to use the text for reference. To encourage him to study models and explanations at the moment when he has hunted them up for himself. To provide an unusually flexible teaching and remedial technique.*

PART III. TABLES

A SECOND COURSE IN ALGEBRA

PART I

THE TEXT

HOW TO STUDY ALGEBRA

1. ***Set apart a definite time*** *for the study of algebra. Budget your time and be systematic in its use.*

2. ***Concentrate and study thoughtfully.*** *Try to understand rather than merely to remember rules and follow them blindly. Education is growing understanding.*

3. ***Take no step which you cannot explain.*** *In explaining an operation tell exactly what steps you take; avoid such words as "cancel," "transpose," and "cross-multiply." Repeat explanations rather than rules.*

4. ***Be accurate.*** *Do not blunder; review each step if it is only the copying of an example. Check everything: in long computations check each step as you take it so that no errors may be carried forward into the later work. Use your judgment constantly to test the reasonableness of your results.*

5. ***Assume responsibility for making up your deficiencies.*** *Let the tests help you to learn; profit by your errors. Discover typical errors and learn how to guard against them; repeat the exercises or study the theory which will strengthen you where you have failed.*

6. ***Adopt the Plan of Problem Solution*** *given on page* 169. *It will help you to form right habits of thought as well as to solve verbal problems. Refer to it wherever necessary.*

7. ***Study the subject matter*** *rather than merely the book. Learn to use the book for reference; become familiar with its index and table of contents. Go to the dictionary or other reference book when necessary. When in doubt about how to proceed, try to reason the matter out, if you do not succeed, look it up.*

8. ***Think well about the " thought questions "*** *proposed by the book or by your teacher. They will help you to understand what algebra can do and how. Approach each new topic with questions in your mind. Disabuse your mind of the idea that algebra is hard or uninteresting or unimportant; learn to understand its fundamental ideas and to know why they are important in this scientific age.*

A SECOND COURSE IN ALGEBRA

CHAPTER I

THE NATURE AND PURPOSE OF ALGEBRA

Some Questions to Have in Mind

1. What is it that makes algebra important in this age of science?

2. Why is algebra usually required for admission to classes in physics and for admission to colleges in which courses in science are taught?

3. What do you understand by the statement that algebra supplies the language of mathematics? What purpose does this language serve? Sentences written in this language are called by what names?

4. The three fundamental notions of algebra as presented in this chapter are *symbols* and their uses, *equations* and their uses, and the *relations* between numbers. Why is each of these important?

Read *How to Study Algebra* on the opposite page.

An Intelligence Test in Algebra

OBJECT: *To show how intelligently you think about your algebra; to bring back to your mind many important ideas of elementary algebra; to show what you need to review; to enlarge your view of what algebra is and what it can do; to help you form at the outset a correct attitude toward algebra and correct habits of study. Remember that your success in passing the course and in getting into college depend upon* (1) *your intelligence and* (2) *your determination and willingness to study.*

PART I. THE MEANING AND USE OF SYMBOLS

It has been said that with the aid of algebraic symbols pupils fifteen years of age can follow reasoning which, without these symbols, could be followed only by the ablest adults.

Most of the ideas involved in Part I of this test are discussed on pages 11–23.

The questions designated by letters instead of numbers are not part of the test. They are intended for class discussion.

1. Write without exponents an expression equivalent to $5a^2b^3$.

2. Write without exponents an expression equivalent to $(a-4)^3$.

(*a*) Tell what purpose an exponent serves.

3. $x^3 \cdot x^4 \equiv x^?$ **4.** $x^3 \div x^2 \equiv x^?$

(*b*) Write examples 3 and 4 without exponents and show that your answers depend upon the meaning of exponents. State the laws of exponents in multiplication and division of terms. Give the meaning of the symbol $\equiv$ and tell how it is different in meaning from the symbol $=$.

According to the laws of exponents:

5. $x^5 \div x^5 \equiv x^?$ **6.** $x^4 \cdot x^0 \equiv x^?$

Reduce to lowest terms:

7. $\dfrac{3a^3b^2}{6ab^4}$

8. $\dfrac{3(x-4)}{(x-4)^3}$

9. $\dfrac{5a(x-2)^2}{10a^2(x-2)}$

(*c*) In examples 7, 8, and 9 did you add, subtract, multiply, or divide? In answering do not use the word *cancel.*

10. Is the value of the fraction $\dfrac{7(5+4)}{7(5+3)}$ changed if the 5's are struck out?

11. Is the value of the preceding fraction changed if the 7's are struck out?

12. The expression $3x - 4 + 4$ can be simplified by uniting, subtracting, or dividing the 4's. Which?

(*d*) In examples 10, 11, and 12 what two processes are illustrated which are sometimes described by the word *cancel?*

13. Multiply by 6 the numerator and denominator of $\dfrac{4\frac{1}{3}}{3\frac{1}{2}}$

In each fraction, 14–17, multiply numerator and denominator by the smallest number which will produce a simple fraction.

14. $\dfrac{5\frac{1}{4}}{4\frac{1}{3}}$ **15.** $\dfrac{\frac{5}{a}}{\frac{6}{ab}}$ **16.** $\dfrac{\frac{7}{ab^2}}{\frac{8}{a^2b}}$ **17.** $\dfrac{x-3}{5-\frac{x}{2}}$

(*e*) Divide $\left(3+\dfrac{a}{b}\right)$ by $\left(\dfrac{b^2}{a}+b\right)$ by indicating the division by writing the first of the expressions as the numerator and the second as the denominator and then multiplying as suggested above.

18. In the expression *abc*, what is the coefficient of *c*?

Rewrite each expression, 19–22, inclosing the coefficient of a in a parenthesis.

19. $3a - ab$ **20.** $ta + ax - a\sqrt{2} - a$

21. $ab - 2ab^2 - b^3$ **22.** $a(b + c) + a(c - d)$

23. Unite $3(a - b) - (a - b) + \frac{1}{2}(a - b)$

(*f*) The coefficient *one* is usually omitted. This may lead a careless pupil into what error? Illustrate in examples 20 and 23.

24. $3(x - y) - c(x - y) \equiv ?(x - y)$

25. $a(x + y) - (x + y) \equiv ?(x + y)$

26. $9(ab)^2 \equiv (?)^2$ **27.** $4\sqrt{ab} \equiv \sqrt{?}$

28. $2 - \dfrac{5 - a}{3} \equiv \dfrac{?}{3}$

In each example, 29–36, unite the two terms into a single term if possible, otherwise write "impossible."

29. $3x + 3x$ **31.** $x^3 + x^3$ **33.** $x_3 + x_3$ **35.** $x'' + x''$

30. $3x + 4x$ **32.** $x^3 + x^4$ **34.** $x' + x'$ **36.** $x + .25x$

Write in algebraic symbols the two phrases below:

37. The square of the difference of two numbers.

38. The difference of the squares of two numbers.

39. Is $\sqrt{a^2 + b^2} \equiv a + b$?

40. Verify your answer to question 39 by substituting 3 for a and 5 for b.

41. $(b - a)$ multiplied by $-1 \equiv ?$

42. $(-a - b)$ multiplied by $-1 \equiv ?$

43. $(-4)(-6)(-3) \equiv ?$ **44.** Is $\dfrac{b - a}{-4} \equiv \dfrac{a - b}{4}$?

(*g*) What multiplication will transform the first fraction of example 44 into the second fraction?

45. $(-1)^3 = ?$

46. What is the value of $(-1)^n$ when n is even?

47. What is the value of $(-1)^n$ when n is odd?

48. What is the numerical difference between $15 \cdot 4 - 3$ and $15 - 4 \cdot 3$?

49. What is the value of $3x^2 - 4x + 3$ when $x = -3$?

50. What is the value of $t^3 - 2t^2 - 3t - 1$ when $t = -\frac{1}{3}$?

(*h*) In your opinion, what is the purpose of algebraic symbols? In what sense is algebra more general in its application than arithmetic?

To find your score or percentage on this part of the test multiply the number of your correct answers by two.

If the preceding questions give rise to many difficulties, pages 11–23 may be studied at this point. On pages 21–23 there are other tests on symbols.

PART II. EQUATIONS

Equations are the sentences of algebra.

(Most of the ideas involved in this test are discussed on pages 23–54.)

Questions designated by letters instead of numbers are not part of the test. They are intended for class discussion.

51. In transforming the equation $8x + 5 = 30$ into the form $8x = 25$, do you add, subtract, multiply, or divide?

52. Answer the preceding question for the transformation of $3x - 7 = 20$ into $3x = 27$.

53. What is the smallest multiplier which will transform the following equation into an equivalent equation containing no fractions?

$$\frac{x}{5} = \frac{x}{12} - \frac{7}{3}$$

54. Answer the preceding question for the equation:

$$\frac{7}{x-7}-\frac{3}{x-3}=\frac{4}{x-4}$$

55. By what will you divide each member of the following equation in order to solve it for x? $ax - bx - x = s$

(i) Solve the equation $\dfrac{3}{x+14}=\dfrac{1}{x}$ and explain each step. Do not use the words *cancel*, *transpose*, or *cross-multiply*, but tell whether you add, subtract, multiply, or divide.

56. Is 5 a root of the equation $(x-5)(x+6)=0$?

57. Is 6 a root of the equation above?

58. Is $-n$ a root of $(x+n)(x-4)=0$?

59. What are the roots of $x^2+2x-35=0$?

60. What are the roots of $\dfrac{x-2}{4}+\dfrac{12}{x}=\dfrac{x}{2}$?

(j) Solve for V_1 the equation $\dfrac{V_2}{V_1}=\dfrac{P_1}{P_2}$ and explain each step. Solve for b the equation $t-2ab=3bc$.

61. Solve for x and y: $3x-3y=18 \qquad x+5y=-6$.

62. Solve for a and b: $\dfrac{a}{3}=11-\dfrac{b}{2} \qquad \dfrac{b}{4}=\dfrac{a}{5}$

(k) Show that x can be eliminated from the equations of example 61 by subtraction and also by substitution. What does it mean to *solve* a pair of linear equations in two unknowns? Use the graph on page 255 to help you in your explanation. If you are familiar with the word *locus* use that also.

63. Translate into algebraic symbols: Two more than two thirds of a certain number equals thirty.

64. Find the number mentioned in the preceding example.

(l) Tell how algebraic symbols helped you in examples 63 and 64 to think out the answer and to keep a record of the work.

65. A dealer sold two small cars and five large cars for \$7850. At the same prices he sold one small and two large cars for \$3250. Write the two equations you would use in finding the cost of each kind of car.

66. Give the answers to the preceding problem.

67. If the same number is added to each term of the fraction $\frac{a}{b}$, the resulting fraction will equal $\frac{r}{s}$. Translate this statement into an algebraic equation.

68. What is the number obtained by solving the equation of the preceding problem?

(*m*) Could you readily find the number called for above without the use of an equation? Show that transforming an equation is reasoning about numbers. Use the formula of example 68 for finding the number which must be added to each term of the fraction $\frac{3}{5}$ so that the resulting fraction shall equal $\frac{5}{7}$; to $\frac{3}{4}$ so that the resulting fraction shall equal $\frac{5}{6}$. Check each result.

69. Solve $3(2x - 1) + 8 = 5(x + 1)$

70. When the root of the equation above is substituted for x, what is the value of each member of the equation?

71. Solve $\dfrac{x-3}{x} = \dfrac{2x}{x-2}$

72. Solve for x: $\dfrac{a-x}{b-x} = \dfrac{b}{a}$

73. Solve $6x = 5y \qquad 7x - 6y = -2$

74. What kind of equation has two roots?

75. Name two other kinds of numerical equations with which you are acquainted.

(*n*) In your opinion what purpose do equations (including formulas) serve?

To find your score or percentage on the second part of the test multiply the number of your correct answers by four.

If the preceding exercises gave rise to many difficulties, pages 23–54 may be studied at this time. Other tests on equations are to be found on pages 52–54.

Part III. Number Relations

(Most of the ideas involved in this test are discussed on pages 54–62.)

Questions designated by letters instead of numbers are not part of this test; they are intended for class discussion.

Given the number n, represent the three quantities below.

76. Three less than twice the number.

77. The amount by which the number exceeds 50.

78. The amount by which two thirds of the square of the number is less than 30.

79. Take any integer, multiply it by 6, add 12, divide by 3, subtract 2, divide by 2, subtract the original number, add 9. The result is 10. Complete the following explanation:

$$n;\quad 6n;\quad 6n+12;\quad 2n+4.$$

80. Take any integer; double it; add 4; multiply by 3; subtract 12; divide by 2. Give me your result and I will tell you the number with which you started. Write an algebraic explanation similar to that above.

81. People who do not know how to express number relations in algebraic symbols, sometimes amuse themselves with number puzzles which they could very easily solve if they knew how to use equations. Consider the following ancient puzzle: "A mule carrying a load of corn met a donkey similarly loaded. The mule said to the donkey, 'If you gave me one measure, I should carry twice as much as you. If I gave you one, we should carry equal burdens.'" State in two equations the relations described.

82. Write the results obtained by solving the equations of 81.

83. If the smallest of three consecutive numbers is n, what are the other two?

84. If the smallest of three consecutive odd numbers is a, what are the other two?

85. Of three consecutive integers the middle one is x. Express the product of the first and last.

86. How many cents are there in q quarters, d dimes, and c cents?

87. What is the cost in cents of m articles bought at h cents a hundred?

88. If the three digits of a number from left to right are 1, 7, and 5, the number is 175. If the three digits from left to right are x, y, z, represent the number.

89. What is the interest for one year on five twelfths of x dollars at 4%?

90. Of \$5000, a part which we will call x is invested at 5%, and the rest at $5\frac{1}{2}$%. Indicate the annual income.

91. What is the supplement of x degrees?

92. The ratio of two numbers is 3, the smaller is c, what is the larger?

93. The ratio of two numbers is 1.7, the smaller is 2, what is the larger?

94. The ratio of two numbers is 1.7, and the larger is 5.1, what is the smaller?

95. The larger of two numbers is 81; the ratio is $\frac{1}{7}$. Give the smaller number correct to two-figure accuracy; that is, to the nearest second figure, or to two significant figures. Notice that this does not mean two figures after the decimal point but two figures in all.

96. One hundred is divided into two parts so that one part is 10 less than twice the other. Express this relation as an equation.

97. What are the numbers obtained by solving the equation of 96?

98. Write an equation which expresses the relation between a and b in the following table:

a	3	4	5	6
b	10	12	14	16

99. In the table above, when $b = 54$, what is a?

100. When a telegram costs 55 cents for the first 10 words and 4 cents for each additional word, what is the cost of a telegram of n words if n is more than 10?

(*o*) Comment on the fact that algebra supplies the other sciences with a convenient way to deal with number relations. Tell something of why the relations between numbers is an important subject for study.

To find your score on this part of the test, multiply the number of your correct answers by four. The number of your correct answers for the entire test gives your score or the per cent correct on the Intelligence Test in Algebra. A score of fifty per cent obtained in three forty-minute periods or approximately two hours probably indicates sufficient ability and training for the profitable study of a second course in algebra.

After completing the intelligence test and discussing those questions which gave difficulty and also the lettered questions, ***a*** to ***o***, study as much of pages 11–62 as the results indicate that you need. Other tests are to be found on pages 71–75.

Concerning Algebra

The three central ideas in this course in algebra are:

1. *The use of symbols.* 2. *The use of equations.* (*Formulas included.*) 3. *The study of the relations between numbers.*

The importance of algebra. The study of algebra if correctly carried on is of great importance. Some of the reasons for this importance are:

1. The present age is the age of science. Science has made our modern civilization possible. If we are to understand the modern world we must know something of science and the scientific method.

2. Science consists of ideas arranged systematically, and algebra is the best illustration of simple ideas systematically arranged; hence its study gives the best opportunity to learn what a science is, and to gain experience in arranging ideas into a system.

3. Algebra is essential to many other sciences.

4. The study of the topics of algebra can be made to illustrate the best methods of thinking even in fields remote from algebra.

Some things that algebra does. Algebra supplies an important part of the language of mathematics and gives the best means we have for the concise statement of general laws and rules about numbers. It often reduces an entire paragraph or a mass of data to a single line. It shortens the explanations of problems and aids in their solution. It helps us to reason about number relations. It is of assistance both in making quantitative studies and in writing the conclusions so that they are easy to understand and to use. Algebra helps to solve such problems as the calculation of an eclipse by an astronomer, the computation of an annuity by an insurance expert, the explanation of the electrical theory that has made the radio possible, and many others.

Part I. Symbols

The symbolism of algebra is a truly remarkable invention or series of inventions. With its aid ordinary minds can grasp and use ideas that otherwise could be used by only the ablest minds.

First principle. *In dealing with symbols first see clearly what the symbols mean.* Are they symbols which tell you to perform some operation? Are they symbols which represent numbers and which are to be operated upon? Second, *operate according to the laws which result from the meaning and nature of the symbols.* For example, $2a + 3a$ is $5a$ because two a's and three a's are five a's; also $2b_1 + 3b_1 \equiv 5b_1$ by similar reasoning; but $a^2 + a^3$ is not a^5 because a^2 means aa or a times a and a^3 means aaa, and since these are not similar terms† they cannot be united into one term.

The meanings of symbols. *Exponents* are considered at length in Chapter III. For the present it is enough to observe that $x^2 \equiv xx$ and $x^4 \equiv xxxx$, hence $x^2 \times x^4 \equiv x^6$; that $x^3 + 4x^3 \equiv 5x^3$; and that $x^5 \div x^2 \equiv x^3$ because $\dfrac{xxxxx}{xx} \equiv x^3$. The exponent of a number tells how many times the number is taken as a factor.

In a term any factor may be considered as the *coefficient* of the other factors; for example, in $5ef$, 5 is the coefficient of ef, $5e$ of f, e of $5f$, etc. Point out the coefficients which are united in making the following additions: $(a + b)x + cx \equiv (a + b + c)x$; $(c + d)x + (c + d) \equiv (x + 1)(c + d)$.

In $a(6\sqrt[5]{4b^3 + c_2})$, the a is the (literal) coefficient of the expression in the parenthesis; the 6 is the (numerical) coefficient of the expression in the radical; the 5 is the index of the radical;

† A term, generally speaking, is that part of an algebraic expression which is between two consecutive plus and minus signs, written or understood. Similar terms are terms which are alike except that their coefficients may be different. It is not important to be able to state these definitions in words, but it is important to be able to recognize terms and to know when they are similar. After you are quite familiar with these two ideas it will be interesting to try to define them in your own words. A monomial is an algebraic expression containing one term. What is a binomial? a trinomial? a polynomial?

it indicates that the fifth root† is to be taken; the 3 is the exponent of b, the coefficient of c_2 is 1; the 2 is a subscript and indicates that this is one of several c's. We might for example represent the corresponding sides of several triangles by c_1, c_2, c_3, etc. Primes are used in the same way; a' is read "a prime" (first), a'' is read "a second," a''' is read "a third," etc.

In the expression $5(a+b)$, the *parenthesis* serves (1) to help describe the number which is to be multiplied by 5, (2) to help give the directions that the sum of a and b is to be multiplied by 5. Explain the difference in meaning of $3a^2$ and $(3a)^2$. What is the difference in value when $a = 4$? The bar is one form of a parenthesis; $6 - \overline{a+3}$ means 6 less the sum of a and 3. Notice that the bar in a fraction and in a root sign serve in the same way; $\dfrac{a+b}{3}$ means that the sum of a and b is to be divided by 3. What is the difference in the value of $\sqrt{4} + 5$ and $\sqrt{4+5}$?

Consider the following symbols which are used to indicate *equality* or the lack of it.

$$5 - 2 \equiv 3$$
$$(x-y)^2 \equiv x^2 - 2xy + y^2$$
$$3x - 5 = 20$$
$$8 \neq 7$$
$$9 > 6$$
$$5 < 8$$

The symbol $\equiv$ is often, though not always, used instead of $=$ in order to indicate that two expressions are identically equal, that is, if letters are involved, the expressions are equal for all values of the letters. The symbol $\neq$ is read, *is not equal to*. The last two symbols are read *is greater than* and *is less than* respectively; they may be thought of as equality signs pinched together on the end next to the smaller of the two numbers.

Three meanings of the minus sign. In algebra the minus sign has historically three meanings:

I. Subtraction, as in $8 - 2 \equiv 6$

II. A shortage, as the second sign in $5 - 9 \equiv -4$

† The fifth root of a number is one of the five equal factors of the number; for example, $\sqrt[5]{32} \equiv 2$.

III. A direction, as on this number scale:

The first meaning became familiar to you in arithmetic. It is, of course, a very ancient one. The second meaning is also old, and it has played a large part in the history of algebra; in fact, the word *algebra* is derived from an Arabic expression which means *making up the shortages* as we do in solving equations. By means of these first two meanings, we can completely explain all the laws of signs as we shall soon see. The third meaning, which of course includes the second, is not so old, having been first comprehended and applied by Descartes (1596–1650).

1. Historically what are the three meanings of the plus sign? (Use the word *over* or *surplus*). What is meant by the statement that a signed number indicates both size and direction? Show that a plus (or a minus) sign can serve (1) as a symbol or operation, and (2) as an aid in representing a number.

2. Which is larger, 3 or -5, and how much larger? The answer is that a *surplus* of 3 is 8 larger than a *shortage* of 5. This question is sometimes confusing to beginners because in *absolute value*† -5 is greater than 3. Explain.

Explaining the Laws of Signs

1. Repeat the "first principle for dealing with symbols." (See page 12.) It is not enough to be able to *state* a law of signs (or any mathematical law). It is interesting and important to be able to give the reason behind the law. The habit of finding such reasons plays a large part in education.

Algebraic addition of monomials.

2. Using the words "short" and "over" or "shortage" and

† The *absolute* or arithmetical *value* of a number is its value without regard to sign. In other words, it is its magnitude without regard to its direction. The absolute value of x is written $|x|$.

"surplus," explain each of the additions indicated below. (For algebraic addition no rule is necessary.)

$$\begin{array}{ccccc} 3 & -4 & 6 & -6 & -5 \\ \underline{5} & \underline{-3} & \underline{-7} & \underline{7} & \underline{5} \end{array}$$

3. $ax + bx - x \equiv (? + ? - 1)x;\quad rs - rt + r \equiv (? - ? + ?)r$

4. Unite the similar terms in $2x + ax - bx$. Express the result in the way suggested by the preceding example. Express as one term $ar - br + r$. In the expression $ab + bx - b$, enclose the coefficient of b in a parenthesis.

Try Exercise 1, page 347, for practice in addition.

Explaining the law of signs for subtraction.

5. Repeat the following explanations until you have mastered them. In the expression $50 - (2a + 3)$, we are told to subtract $2a + 3$; when we write $50 - 2a$ we have not subtracted enough, we must subtract 3 more and write $50 - 2a - 3$ or $47 - 2a$. In the expression $30 - (4a - 5)$, we are told to subtract 5 *less* than $4a$, and when we write $30 - 4a$, we have subtracted 5 too much; we must put back the 5 and write $30 - 4a + 5$ or $35 - 4a$. Notice that in subtracting $+3$ we wrote -3, and that in subtracting -5, we wrote $+5$ and then united. Notice that subtracting a shortage of 5 is equivalent to adding 5.

6. Subtract, and explain each subtraction. Check by addition.

$$\begin{array}{cccccccc} 1 & -5 & -3 & 7 & 8 & -4 & -5 & -5 \\ \underline{2} & \underline{6} & \underline{-4} & \underline{-8} & \underline{-7} & \underline{-3} & \underline{-5} & \underline{+5} \end{array}$$

Rule for the subtraction of signed terms: *Read the subtrahend with the sign changed and then proceed as in addition.*

Perform the subtractions indicated in examples 7–14. Notice that the subtractions are indicated by means of minus signs and parentheses; when we are *subtracting*, we follow the rule for subtraction; the parentheses themselves have nothing to do with the changes of sign.

7. $5 + 6a - (8 + a)$

8. $6 - 7a - (9 - a)$

9. $3x - 5 - (3x - 5)$

10. $x + 4 - (2x - 5)$

11. $3a^2 - (a^2 - a)$

12. $a + b - (a + b)$

13. $a + b - (a - b)$

14. $.5x - (2 - x)$

Try Exercise 2, page 350, for practice in subtraction.

15. Explaining the law of signs for multiplication. First consider the meaning of multiplication. 3×4 means $3 + 3 + 3 + 3$ and -3×4 means $-3 + (-3) + (-3) + (-3)$. It is impossible to give any such meaning to $-3(-4)$. Why? We can, however, explain multiplication by a negative number if we think of it as a multiplication *and* a subtraction. In $(-3)(-4)$ the multiplication of -3 by 4 gives -12, and the subtraction gives 12. In the expression $x + 5(y - 7)$, we are told to multiply $(y - 7)$ by 5 and to add the result to x. The multiplication gives $x + (5y - 35)$ and the addition gives $x + 5y - 35$. In the expression $a - 3(b - 4)$, we are told to multiply $(b - 4)$ by 3 and to subtract the result from a. The multiplication gives $a - (3b - 12)$, and the subtraction gives $a - 3b + 12$. Evidently multiplication by -3 is equivalent to a *multiplication and a subtraction*: $3 \times b$ gives $3b$ and the subtraction gives $-3b$; 3 times "short" 4 is "short" 12, and the subtraction gives $+12$. Perform the following multiplications and explain each one. When the multiplier is negative, explain the multiplication as a multiplication and a subtraction.

$$\begin{array}{r} -3 \\ 2 \\ \hline \end{array} \qquad \begin{array}{r} 4 \\ -5 \\ \hline \end{array} \qquad \begin{array}{r} 3 \\ 8 \\ \hline \end{array} \qquad \begin{array}{r} -4 \\ -7 \\ \hline \end{array}$$

Rule for multiplication of signed terms: *If two factors have like signs their product is positive; if two factors have unlike signs their product is negative.*

16. Multiply: $(-1)(a + b)$ $\quad (-1)(a - b)$ $\quad (-1)(-1)$ $\quad (-1)(-1)(-1)$

17. In the following fractions multiply each numerator and each denominator by -1.

$$\frac{b - a}{-4} \qquad \frac{-25}{-150} \qquad \frac{5}{y - x} \qquad \frac{a + b}{-2}$$

18. Simplify the following fraction by multiplying numerator and denominator by 6:

$$\frac{x - \frac{y}{6}}{x - \frac{2}{3}}$$

19. *A zero factor.* Multiply $3(-4)(0)(-5)$. Principle: *The product of several factors is zero when, and only when, one of the factors is zero.*

Try Exercise 3, page 350, for practice in multiplication.

20. Explaining the law of signs for division. Division is the inverse † of multiplication, hence the law of signs for division follows from that for multiplication. It is this: *If two terms have like signs their quotient is* ——; *if they have unlike signs,* ——.

Divide, and check by multiplication:

$$\frac{7x}{x} \qquad \frac{-6x}{3} \qquad \frac{-8a}{2} \qquad \frac{9a}{-3a} \qquad \frac{x}{-x} \qquad \frac{-12(a-b)}{-3(a-b)}$$

21. Find the value of:

$$\frac{-8}{-2} \qquad \frac{8}{2} \qquad \frac{-8}{2} \qquad -\frac{8}{-2}$$

22. If $5x = 0$, what is the value of x? $\frac{0}{7} = ?$ What in your opinion is the value of $\frac{7}{0}$? Can you give any meaning to division by zero?

In making algebraic transformations, do not divide by zero.

Study the following transformations and notice the absurd result which was produced by the division of ③ ‡ by zero.

† The inverse of an operation is the operation which is destroyed by the original operation.

‡ ③ is read "equation three."

Let	① $x - 2 = 0$	
then	② $3(x - 2) = 0$	① $\times 3$
	③ $5(x - 2) = 2(x - 2)$	② $+ 2(x - 2)$
	④ $5 = 2$	③ $\div (x - 2)$

Try Exercise 4, page 351, for practice in short division.

The Order of Operations

1. The expression $4 + 5 \times 2$ is ambiguous until we agree which operation to perform first. Why? In finding the value of such expressions it is agreed to *perform first the multiplications and divisions in the order in which they occur, and then to perform the additions and subtractions in any order.* What is the value of the expression above? When multiplication is indicated by juxtaposition, we assume that the multiplication has already been performed. For example,

$$9\,a \div 8\,x^2 \equiv \frac{9\,a}{8\,x^2} \text{ but}$$

$$9\,a \div 8 \times x^2 \equiv \frac{9\,a}{8}\,(x^2).$$

Explain. Show that the rule: *Simplify terms before uniting,* covers nearly the same ground as the agreement above.

2. What is the value of $4 \times 3 - 3$? of $4 \cdot 8 \cdot 2$? of $3 \cdot 0 + 4$? of $3 \cdot 4 + 2 - 1$? of $8\,x \div x + 2(-1)$? of $10x^3 \div 2\,x + 6x \div 2 \times x$?

3. Find the value of each of the following expressions. In which of them is it more convenient to unite the terms within the parentheses (or joined by the bar) before performing the indicated multiplications and divisions?

$$\frac{8+12}{4} \qquad \frac{8 \times 12}{4} \qquad \frac{4 \times 5}{3} \qquad 7(5-6), \qquad \frac{15}{8-5}$$

Try Exercise 5, page 352, for practice in operating in the correct order.

Substitution and Evaluation

1. Tell what you can of the importance of substitution and evaluation in the study and use of algebra.

2. Restate the "first principle" for dealing with symbols.

3. Does the expression $\frac{6x^2}{5}$ have a constant value or may its value vary as x varies? Upon what is its value dependent? What is the value when $x = 1$? when $x = \frac{1}{3}$? when $x = \frac{1}{2}$?

4. If $a = 6$, $b = -17$, and $c = 12$, find the value of

$$\frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

By substituting 4 for a and -3 for b, test the following identities:

5. $5a^2 \overset{?}{\equiv} (5a)^2$

6. $-(2a)^2 \overset{?}{\equiv} (-2a)^2$

7. $(a+b)^2 \overset{?}{\equiv} a^2 + 2ab + b^2$

8. $(a-b)^2 \overset{?}{\equiv} a^2 - 2ab + b^2$

9. $(a+b)(a-b) \overset{?}{\equiv} a^2 - b^2$

10. $\sqrt{(a-b)^2} \overset{?}{\equiv} a - b$

11. $\sqrt{a^2+b^2} \overset{?}{\equiv} a + b$

12. $\sqrt{16a} \overset{?}{\equiv} 4\sqrt{a}$

13. If $x = \frac{4}{5}$ does $x + \frac{x}{x-1} = \frac{(x+4)(x-2)}{x+1}$?

14. If $x = y - 2$, what is the value of $ax - bx$?

Try Exercise 6, page 352, for practice in evaluating.

A Study in the Use of Algebraic Symbols

***1.** Required to prove that a common factor of two algebraic expressions is a factor of their sum and of their difference.

First decide by trial whether the theorem is true for selected pairs of small numbers. Next observe that you can never *prove* the theorem by numerical illustration. Finally see how easy

* Starred exercises may be omitted in a minimum course; see *A time schedule*, page viii.

it is to give a conclusive proof if algebraic symbols are used. Thus:

(1) Let na and nb represent any two algebraic expressions having a common factor n.

(2) Then $n(a + b)$ will represent their sum and

(3) $n(a - b)$ or $n(b - a)$ will represent their difference.

(4) Both the sum and the difference are divisible by n, which was to be proved.

*2. Discuss advantages derived from the use of algebraic symbols.

Tests of Your Understanding and Skill.† Symbols

Test A. To Test Your Understanding

1. Explain the laws of signs for subtraction and multiplication of signed numbers. (Use the words *short* and *over* or *shortage* and *surplus*.)

2. Would you prefer to have a mind which is satisfied to state rules and apply them or a mind which looks deeper and asks for reasons for the rules? Why?

3. Mention at least two things which can be done more easily with algebraic symbols than without them.

4. Give at least three methods for guarding against errors in your work in algebra.

Test B ‡

1. Add:
$$\begin{array}{r} -6x \\ \underline{-5x} \end{array} \qquad \begin{array}{r} -6 \\ \underline{10} \end{array} \qquad \begin{array}{r} -6 \\ \underline{0} \end{array} \qquad \begin{array}{r} 6m \\ \underline{m} \end{array} \qquad \begin{array}{r} (a-b+c)y \\ \underline{(b-c-a)y} \end{array}$$

† Read paragraphs 3 and 4 of *How to Study Algebra*, page xx.

‡ Students who make poor records on Test B should repeat the exercises indicated by their errors and then try Test C.

2. In the preceding example, subtract each lower term from the number above it.

3. From the sum of $4(a+b)$ and $-8(a+b)$ subtract the sum of $-5(a+b)$ and $-3(a+b)$

4. Unite similar terms: $5\,ab - 11\,ab + ab$; $ax - bx + cx - x$

5. Multiply:

$$\begin{array}{r} 4 \\ \underline{-3} \end{array} \qquad \begin{array}{r} -4\,x^2 \\ \underline{-5\,x^2} \end{array} \qquad \begin{array}{r} -3\,a^2x^2 \\ \underline{2\,a^2x} \end{array} \qquad \begin{array}{r} 3\,ax^2 \\ \underline{0} \end{array} \qquad -5\,ax(1-a+x)$$

6. Divide as indicated; express answers in simplest form; check by multiplication:

$$\frac{10\,b}{5\,b} \qquad \frac{-12\,m^5}{-6\,m^2} \qquad \frac{-5(a-b)^3}{(a-b)} \qquad \frac{0}{a} \qquad \frac{1.2\,a^2-3\,a^4}{.6\,a^2}$$

What is the value of:

7. $(-8) \div 2 + 3(-2)$? 8. $(5-3)(-8) \div (-4)(-10)$?

9. If $a = -2$, what is the value of

$$a^3;\ 2\,a^4;\ -a^2;\ (-a)^2;\ (2\,a)^2?$$

10. When $a = 3$ and $b = -4$, is $\dfrac{a^3-b^3}{a-b} \equiv a^2+ab+b^2$? Prove that the equality is true for all values of a and b $(a \neq b)$.

Test C

1. Add:

$$\begin{array}{r} -4 \\ \underline{3} \end{array} \qquad \begin{array}{r} -3\,x \\ \underline{9\,x} \end{array} \qquad \begin{array}{r} 0 \\ \underline{4} \end{array} \qquad \begin{array}{r} 9\frac{3}{4} \\ \underline{-6.5} \end{array} \qquad \begin{array}{r} 7.8 \\ \underline{-3\frac{1}{2}} \end{array}$$

2. In the preceding example subtract each upper number from the number below it.

3. From the sum of $3(a-b)$ and $-4(a-b)$ take the sum of $-5(a-b)$ and $2(a-b)$

4. Unite similar terms: $-\frac{1}{6}\pi r^2 - \frac{2}{3}\pi r^2$; $at - bt - ct - t$

5. Multiply:

$$\begin{array}{r} -5 \\ \underline{7} \end{array} \qquad \begin{array}{r} -4x \\ \underline{-3x^2} \end{array} \qquad \begin{array}{r} -3ay^2 \\ \underline{2ay} \end{array} \qquad \begin{array}{r} 4ax^2 \\ \underline{0} \end{array} \qquad -4b(1-a-b)$$

6. Divide as indicated; express answers in simplest form; check by multiplication:

$$\frac{12a}{18a} \qquad \frac{-15m^4}{-3m} \qquad \frac{-5(a-b)^3}{(a-b)} \qquad \frac{a}{2a^2} \qquad \frac{4\frac{1}{2}a^2 - 3a}{1\frac{1}{2}a}$$

What is the value of:

7. $(-15) \div 3 \cdot 4 + (-9)$? **8.** $(8-2)(-3) \div (-1) + (-2)$?

9. If $a = -2$, what is the value of $3a^2$? $(3a)^2$; $-a^3$; $(-a)^3$; $2a^3$?

10. When $a = 1$ and $b = -2$, is $\frac{a^3+b^3}{a+b} \equiv a^2 - ab + b^2$?

Prove that the equality is true for all values of a and b $(a \neq b)$.

Test D

1. Is $x^2 + x^3 \equiv 2x^5$? $x^6 \equiv x^2 \cdot x^3$? $b_2 + b_2 \equiv 2b_4$? $b^2 + b^2 \equiv b^4$?

2. From the sum of $3x^3 + 2x^2 + ax + 1$ and $4x^3 - 2x^2 + ax$, take $5x^3 - x^2 + 3 - b$.

3. Does $19x - \frac{2(x^2+5)}{x-1}$ equal zero if $x = -\frac{2}{3}$?

4. What is the value of $4 - 5 \cdot 3 + 2 + [4(-5) \times (3+2)]$?

5. What is the value of $8 - \frac{3 + 2 - 5 \cdot 6}{(-1)(3+2)(5 \cdot 6)}$?

Test E

1. Simplify by multiplying numerator and denominator by a suitable multiplier:

$$\frac{4\frac{2}{3}}{7\frac{1}{2}} \qquad \frac{x}{5+\frac{x}{4}} \qquad \frac{x-5}{x+\frac{2}{3}} \qquad \frac{a+\frac{b}{c}-\frac{c}{d}}{\frac{a}{c}+\frac{1}{d^2}}$$

2. What is the effect of multiplying an expression by -1 twice in succession? Of multiplying by -1 followed by division by -1? Of multiplying both a fraction and its numerator by -1? Illustrate each answer.

3. Which of the following are identities?

$$\frac{a}{-b} \overset{?}{\equiv} \frac{-a}{b} \overset{?}{\equiv} -\frac{a}{b} \qquad \frac{b-a}{-c} \overset{?}{\equiv} \frac{a-b}{c} \qquad \frac{b-a}{-c} \overset{?}{\equiv} \frac{b+a}{c}$$

$$\frac{b-a}{c+n} \overset{?}{\equiv} \frac{a-b}{c+n} \qquad \frac{c-d}{e-f} \overset{?}{\equiv} \frac{d-c}{f-e} \qquad \frac{g-h}{a-b} \overset{?}{\equiv} \frac{g+h}{a+b}$$

The ability to answer the questions of Part I of the intelligence test on page 2 and Tests A to E on pages 20–23 indicates sufficient mastery of the preceding pages and the accompanying exercises.

Part II. Equations

Questions to Keep in Mind as You Study

1. What purposes do equations serve?

2. What is the difference between an identity and an equation of condition?

3. What is the fundamental law for the transformation of equations? What are the advantages of setting down "directions" when transforming equations?

4. How can the word *transpose* be replaced by an exact statement of the operation performed?

5. What three types of equations are studied in this chapter?

6. What is the "first principle for the use of symbols"?

Concerning Equations

1. Second principle for the use of symbols. Algebraic equations are written in symbols, and in equations we find the most important uses of the symbols. This idea brings us to the second principle for the use of symbols, namely, *More important than the symbols themselves are the ideas for which the symbols stand and the purposes which the symbols serve.*

2. The uses of equations. Equations serve (1) to state facts or rules briefly (consider formulas, for example); (2) to explain the solution of problems; (3) to keep a concise record of a mathematical proof or argument by which a result is reached.

3. Two kinds of equations.

1. The equation $(x-3)^2+3(x-4)^2=4(x-5)^2-3$ suggests a question. *It is a conditional equation.* The answer to the question, or the *root* of the equation, is 4. The condition on which the two members of the equation are equal is that $x=4$. Prove it. (A principle to be followed in checking equations is stated on page 27.)

2. Classify the following as *identities*, true for all values of the letters; or as *equations of condition*, true only for certain values of the letters:

(*a*) $x^2-4x=x(x-4)$; (*b*) $x^2+1=x(1+x)$;
(*c*) $(a+b)^2=a^2+2ab+b^2$; (*d*) $x-6=2x-9$;
(*e*) $\dfrac{ab+b}{b}=a+1 \quad (b \neq 0)$.

4. The transformation of equations. The fundamental law for the transformation of equations is: *If you change the value of one side of an equation, make the same change in the value of the other side.* You may change the values by adding the same num-

ber to both sides, subtracting the same number from both sides, multiplying both sides by the same number, or dividing both sides by the same number (not zero). You may change the *form* of either side without changing its value, as when you perform indicated multiplications, or substitute, or unite similar terms.

The fundamental habit to form is: *Decide exactly what change in value you are to make; make it carefully and accurately.* To set down, at the right, directions for making these changes in value is a means of preventing errors and of making it easier for yourself or any one else to review your work. A slogan for mathematicians is: A good mathematician leaves a trail when he works which any other mathematician can follow.

5. Equivalent equations. When an equation is transformed in accordance with the fundamental law stated above, each new equation is *equivalent* to the original equation, that is, it is satisfied by the same root or roots. (Certain exceptions will be noted later.) Are the following pairs of equations equivalent?

(a) $\begin{cases} x + 3 = 6 \\ x = 3 \end{cases}$ (b) $\begin{cases} 3x - 2y = 27 \\ 15x - 10y = 135 \end{cases}$

(c) $\begin{cases} x - \dfrac{3 - x}{5} = 15x \\ 6x - 3 = 75x \end{cases}$ (d) $\begin{cases} x - 3(2x - 5) = 18 \\ -5x - 15 = 18 \end{cases}$

6. Tell how the first of the equations below is transformed into the second.

$$\frac{3x - 5}{2} = \frac{4x - 7}{12}$$

$$18x - 30 = 4x - 7$$

Tell how the following fraction is transformed from the first form into the second.

$$\frac{3x - 5}{2} \equiv \frac{18x - 30}{12}$$

Why is there a denominator in the second instance and not in the first?

Linear Equations in One Unknown

1. A study of the solution of equations. (Re-read paragraph 3 of *How to Study Algebra,* page xx.) Consider the following equations and the transformations required for their solution:

I. ①[†] $3x - x - x = 8 - 5$
② $x = 3$

Step I. Unite similar terms in each member. Since there is no change in values no directions need be written.

II. ① $2x - 8 = 4 - x$
② $3x = 12$ ① $+ x + 8$

Step II. If there are any terms with minus signs before them, add enough to make up these shortages. (It is interesting to recall that the word *algebra* comes from an Arabic word meaning *restoration* or *making up the shortages.*)

III. ① $5x = 8 + 4x$
② $x = 8$ ① $- 4x$

Step III. Subtract from each side the smaller unknown term.

IV. ① $x + 3 = 8$
② $x = 5$ ① $- 3$

Step IV. Subtract from each side any known term which stands beside an unknown term.

V. ① $6x = 12$
② $x = 2$ ① $\div 6$

Step V. Divide each side by the coefficient of the unknown term.

The Rule of Diophantus.

2. The five steps[‡] illustrated in the preceding paragraph constitute a systematic procedure for the solution of equations. Applied as required and in the order given, they solve equations by removing one complication at a time. Together they are called the Rule of Diophantus (third century A.D.).

† The symbol ① is read "equation one."

‡ They were first stated in their present form by George W. Evans, of Boston. See *Algebra for Schools* (N.Y. 1899), p. 15.

Explain each step of the following solution:

①	$3x + 2(2x - 9) = x - 12$	
②	$3x + 4x - 18 = x - 12$	The equation is now in form for the application of the rule.
③	$7x - 18 = x - 12$	Has the value of either member been changed?
④	$7x = x + 6$	③ + 18. What change in value has been made?
⑤	$6x = 6$	④ $- x$
⑥	$x = 1$	⑤ $\div 6$

Ans: 1 Check:

$$3 + 2(-7) \stackrel{?}{\equiv} 1 - 12$$
$$3 - 14 \quad \Big| \quad -11$$
$$-11 \quad \Big|$$

(In checking an equation, never change the value of either member.)

Solve and check the following equations. Explain each change in values. In examples 5–8, perform the indicated multiplications before applying the rule.

3. $5x - 10 = x + 14$

4. $12x - 5 = 4x + 12 + 3$

5. $17x + 5(2 - 3x) = 18$

6. $7(3 - x) - 4(7 - 2x) = 0$

7. $3(x - 6) - 2(4 - x) = 0$

8. $6x - 2(x + 3) = 5(2x - 6)$

9. Abbreviating the rule. Steps III and IV may be performed at the same time by any careful pupil. Some pupils may be able to combine steps II, III and IV into one step; however, it pays at first to work slowly and accurately.

Solve: $4(x - 2) + 3(2 - x) - 3x = 6(x + 1)$

10. Try to solve $3(x - 4) + x = 4(x - 3)$. Explain the result. What kind of equation is this?

Try Exercise 7, A, page 354.

11. Equations containing monomial denominators.

$$\text{①} \quad \frac{5y}{6} - \frac{21}{5} - \frac{5y-2}{10} = 0$$

$$\text{②} \quad 25y - 126 - (\qquad) = 0 \qquad \text{①} \times 30$$

Complete the solution and check the result.

Try Exercise 7, B, page 354.

12. Literal equations. Solve for y.

① $ay - 2ab = 4ab - ay$

② $2ay = ?$ ① $+ ay + 2ab$

③ $y = ?$ ② $\div 2a$

Check by numerical substitution or by substituting the root in the original equation. In substituting numbers avoid 1 and 0.

13. Solve for x, $ax + bx = 3 + cx$

14. The same number is added to each term of the fraction $\frac{c}{f}$ and the resulting fraction equals $\frac{h}{k}$. What is the number? Use the equation $\frac{c+x}{f+x} = \frac{?}{?}$. See example 68, page 7. The discussion there suggests two important ideas, one about symbols and one about transforming an equation. What are they?

15. In a certain theorem in geometry it is required to find A in terms of x and y; and it is known that $B = A + C$, $B = \frac{1}{2}x$, and $C = \frac{1}{2}y$. Make the necessary transformations and substitutions and express the answer as a formula. If $x = 57°$ and $y = 32°$, find A. The ability to use algebraic symbols and to transform equations is very useful in the study of geometry.

Try Exercise 8, page 355. If necessary repeat Exercise 1, B. It will help you in finding the coefficients of the unknowns.

Linear Equations in Two Unknowns

1. What is a linear equation? (See page 253.) What does it mean to solve a linear equation in one unknown? How do you know when the solution is correct?

2. Can a linear equation in two unknowns be solved in the sense in which the word solved is used in the preceding example? Consider, for example, the equation $2x + 3y = 3$; if *any* number is substituted for one unknown, the resulting equation can be solved for the other unknown; for *each* value of x there is a *corresponding* value of y. This means that there are how many pairs of values which satisfy the equation? Likewise an indefinite number of pairs of values will satisfy the equation $x - y = 4$; but there is only *one* pair of values which will satisfy *both* of these equations. The graph on page 255 will help to make this clear. *To solve a pair of linear equations in two unknowns is to find a pair of values of the unknowns which satisfy both of the equations.*

3. Systems of equations in more than one unknown are sometimes called simultaneous equations. Can you tell why?

4. Solve and check:

①	$3x + 4y = -6$		
②	$5x - 3y = 19$		
③		① × 3	What is accomplished by these
④		② × 4	two steps?
⑤		③ + ④	What is accomplished by this step?
⑥	$x =$	⑤ ÷ 29	
⑦	$6 + 4y = -6$	⑥ substituted in ①.	
⑧		⑦ − 6	

Ans.
$x = ?\quad y = ?$

Check: In checking, why is it necessary to substitute the values in *both* of the original equations?

5. ① $\frac{2}{x} - \frac{3}{y} = \frac{4}{3}$

② $\frac{1}{x} + \frac{1}{y} = -\frac{1}{6}$

③ ② × 3

④ ① + ③

Solve as suggested, before clearing of fractions. Solve also by clearing of fractions at the outset.

6. $\frac{x+y}{2x} - \frac{y}{x} = \frac{1}{12}$

$\frac{3x-y}{2} = \frac{y}{4} + 1\frac{1}{8}$

7. $x + y = a$

$x - y = b$

8. ① $ax + by = r,$

② $2ax + 3by = 4.$

③ ① × 2

④ ② − ③

Try **Exercise 9,** page 355.

Quadratic Equations

A quadratic equation in one unknown is an equation involving the second, and no higher, power of the unknown.

***1.** Look up the derivation of the word *quadratic.* It is related to the word *quadrature* in "the quadrature of the circle," which was one of the famous problems of the ancients.

2. Solve the equation $10x - x^2 = 16$.

Step I. Rewrite with one member equal to zero.

② $0 = x^2 - 10x + 16$.

Step II. Factor.

③ $0 = (x - 8)(x - 2)$. Observe that $(x - 8)(x - 2) \equiv x^2 - 10x + 16$

Step III. Set each factor equal to zero and solve each of the resulting equations. (See the principle stated in example 19, page 17.)

④ $x - 8 = 0$ Hence $x = 8$
⑤ $x - 2 = 0$ Hence $x = 2$

Observe that the equation has *two* roots. Check by substituting them in turn in the original equation. In this connection study the graph of this equation on page 256. It crosses the x-axis at what two points?

The review needed. Quadratic equations are often solved by means of factoring, hence we now need to review the kinds of factoring involved; and since factoring is the inverse of multiplication, we shall also review multiplication. You will find that the ability to factor is occasionally helpful in algebra just as it is in arithmetic.

Multiplication of polynomials. Factoring.†

3. Consider these multiplications:

$$\begin{array}{l}x - 3\\ \underline{x - 7}\\ x^2 - 3x\\ \underline{\quad - 7x + 21}\\ x^2 - 10x + 21\end{array} \qquad \begin{array}{l}x - a\\ \underline{x - b}\\ x^2 - ax\\ \underline{\quad - bx + ab}\\ x^2 - (a + b)x + ab\end{array}$$

The x^2's, the 21, and the ab are sometimes referred to as *straight products* and the $-3x$, $-7x$, $-ax$, and $-bx$ as *cross products*. Why? In each product the first term is a square; the third term is the product of . . . .; and the coefficient of the second term is the algebraic sum of . . . . .

4. Multiply $(x - 8)(x + 3)$; $(y - b)(y + c)$; $(a + b)(a - 3)$.

5. Multiply $(x + 2)(x + 2)$; $(x - 3)^2$; $(a + b)^2$; $(a - b)^2$. In each of these products are the first and last terms squares? Why? Is each middle term a double, that is, the sum of two equal terms? Explain. Tell how to recognize a perfect trinomial square. The square of a binomial contains how many terms?

6. Square $x - 8$; $x + y$; $x + 3y$; $5a - 3b$; $r - s$.

7. Multiply $(x - 5)(x + 5)$; $(x - a)(x + a)$. In each product what is the coefficient of the middle term? On what condi-

† For a classification of important type products see page 204.

tion does the multiplication of two binomials give a binomial? Which term in the product has the minus sign?

8. Multiply as indicated: $(x+8)^2$; $(x-8)^2$; $(x-8)(x+8)$; $(s+t)^2$; $(s-t)^2$; $(s+t)(s-t)$; $(x+y)(y-x)$; $(2a-b)(2a+b)$.

9. Complete the following statements and illustrate them: The square of the sum of two numbers equals the square of the first plus....... The square of the difference of two numbers equals the....... The product of the sum and the difference of two numbers equals the difference.......

10. What is the product of $(x+3)(x+7)$? Change the numbers and the signs so that the product shall be: (1) a perfect trinomial square with no minus sign; (2) a perfect square with one minus sign; (3) a binomial.

Try Exercise 10, page 357.

Factoring quadratic expressions of the types $x^2 \pm ax \pm b$, $x^2 \pm 2bx \pm b^2$, and $x^2 - b^2$.

11. Factor $x^2 - 2x - 15$.

Step I. $(x \quad)(x \quad)$

Step II. Find all the pairs of factors of the third term; in this case they are $1, -15$; $-1, 15$; $-3, 5$; and $3, -5$. Select to fill the blanks the pair of factors (and the signs) for which the algebraic sum equals the coefficient of the middle term; in this case 3 and -5.

Observe that if the sign of the third term in the trinomial is minus, the signs of the last terms in the factors will be different and that the numerically larger one will have the sign of the .... term.

Step III. Check by multiplication.

Notice that the second and third types mentioned in the heading are special cases of the first type.

12. Factor: $x^2 - 11x + 24$; $x^2 - 4x - 12$; $x^2 + 14x + 49$; $x^2 - 49$; $121 - 4y^4$.

13. Which of the following fractions can be reduced to lower terms by dividing numerator and denominator by the same factor?

$$\frac{4+9}{2+3} \qquad \frac{25x-9}{5x-3} \qquad \frac{a^2+b^2}{3(a+b)} \qquad \frac{a^2-b^2}{5(a-b)} \qquad \frac{a^2-b^2}{2(a+b)}$$

$$\frac{x^2-y^2}{2(x-y)} \qquad \frac{x^2-16}{x+4} \qquad \frac{x^2-25}{x-5} \qquad \frac{36-a^2}{6-a} \qquad \frac{49+b^2}{7+b}$$

$$\frac{49-25a^2}{5a-7} \qquad \frac{a^2-4a-21}{a+7}$$

Try Exercise 11, page 358.

14. Explain each step in the following multiplication:

$$\begin{array}{rrr} 2x & -\ 3y & \\ 3x & +\ 5y & \\ \hline 6x^2 & -\ 9xy & \\ & +\ 10xy & -\ 15y^2 \\ \hline 6x^2 & +\ \ xy & -\ 15y^2 \end{array}$$

Change the coefficients and the signs in the factors so that the product shall be (*a*) a perfect trinomial square with no minus signs; (*b*) a perfect trinomial square with one negative sign; (*c*) a binomial.

15. Multiply: $(a-b)(a^2+ab+b^2)$; $(a-b)^3$

Try Exercise 12, page 359.

16. Factor $6x^2+xy-15y^2$. Compare with example 14.

17. Factor $12x^2-7x-10$. The result will assume one of these three forms:

$(x \quad)(12x \quad)$ $\qquad (2x \quad)(6x \quad)$ $\qquad (3x \quad)(4x \quad)$.

Why? The blanks must be filled with the factors of . Select the pair of factors of 10 and the signs which make the algebraic sum of the cross-products equal to the middle term. Another method is suggested on page 206.

Try Exercise 13, page 360.

18. Solve and check. If necessary, review example 2, page 30.

$$① \ \frac{9}{x+2}+\frac{7}{x}=\frac{20}{x+3}$$

② $9x^2 + 27x + 7x^2 + 35x + 42 = ? + ?$

② = ① × $(x+2)(x)(x+3)$

19. $$\frac{3b+1}{2} - \frac{5b+1}{3b-5} = \frac{3b+2}{4}$$

Plan of solution: First multiply by 4 and simplify; then multiply by $3b - 5$. Contrast this plan with multiplication by $4(3b-5)$.

Try **Exercise 14,** page 360.

The Solution of Quadratic Equations by Formula

20. The roots of the *general quadratic* $ax^2 + bx + c = 0$ are:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

that is, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, which is a formula for the solution of *any* quadratic equation. Notice once more how algebra is able to make *general* statements and to give *general* solutions. The development of the formula is given on page 282. Give the meaning of each letter in the formula.

21. Solve by formula:

① $13x^2 - 7x = 6$

② $13x^2 - 7x - 6 = 0$ The equation is now in form for solution by formula.

③ $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Formula.

④ $a = 13, \quad b = -7, \quad c = -6$ Explain.

⑤ $x = \dfrac{7 \pm \sqrt{49 + 312}}{26} = \dfrac{7 \pm \sqrt{361}}{26} = \dfrac{7 \pm 19}{26}$ Values of ④ substituted in ③

⑥ $x = 1$ or $\dfrac{-6}{13}$ Check both roots.

A method of extracting a square root is suggested in examples 23 and 26 below.

Try Exercise 15, page 362.

Approximate Answers to Quadratic Equations

22. Solve by the formula and give the answer to the nearest hundredth:

① $x = \dfrac{8x+3}{5x}$

② $5x^2 = 8x + 3$ ①$\times 5x$

③ $5x^2 - 8x - 3 = 0$ ②$-8x-3$

④ $x = \dfrac{8 \pm \sqrt{124}}{10}$

⑤ $x = \dfrac{8 \pm 11.14}{10}$ Notice that 11.14 is the square root of 124 to the nearest fourth figure.

⑥ $x = 1.91$ or -0.31 Check both answers. If either check is not exact, explain the discrepancy.

It is evident from ⑤ that if we are to find approximate answers to quadratic equations we must be able to find approximate square roots of numbers.

Finding Square Roots of Numbers †

23. Exact square roots. The square root of a number is one of the two equal factors of the number. Find by inspection the square roots of the following numbers. Verify the results by

† The teacher can of course teach the algebraic formula for square root if he sees fit. The pupil will readily understand the steps which must be taken to find the (already known) square root of $a^2 + 2ab + b^2$, and how he can take the same steps in finding the square root of a number, provided that he makes due allowance for the decimal notation. In order to find a root to the nearest fourth figure he will need to find *five* figures of the root. This method is not illustrated here because it has little or no practical utility. Computers find square roots by one of the methods illustrated here, or by means of a slide rule or a table of logarithms. The College Entrance Examination Board requires "a process for finding the square root of a number, but no process for finding the square root of a polynomial."

multiplication or by division: 36, 3600, 361, 25, 2500, 225, 2.25, 4, 400, .04, 49, 4900, .49.

24. Find the square root of 2209.

Plan of work: 2209 is between 40^2 and 50^2. Explain. If 2209 is a perfect square, its second digit is either 3 or 7. Why? Find the square root and verify.

25. Find the square roots of the following square numbers: 22.09; 42.25; 56.25; 6561; 10404; .5329.

While studying this course in algebra, you should materially increase your ability and judgment in numerical computation.

Try Exercise 16, A, page 362.

26. The rule of Heron. Either exact or approximate square roots can be found quickly by the rule of Heron (about 100 B.C.), which is also called the method of estimate and average. It is a scientific procedure for approximation, which makes you independent of the use of tables, and which enables you to work rapidly if you will take pains to develop the necessary number judgment. Find $\sqrt{349}$ to the nearest third figure.

Step I. *Estimate the square root.* It lies between 10 and 20 and apparently much nearer to 20. Why? It is larger than 15 because 15^2 is 225. A satisfactory estimate, then, is 18.0.

Step II. *Divide by the estimated square root,* extending the division to the nearest third figure. $349 \div 18.0 = 19.4$. (If the quotient equals the divisor, it is the square root. Why?)

Step III. Find the average of divisor and quotient, because one of these numbers is too large and one is too small, and the square root is approximately midway between them.

$$\tfrac{1}{2}(18.0 + 19.4) \equiv 18.7 \text{ or } 18.0 + \tfrac{1}{2}(19.4 - 18.0) \equiv 18.0 + .7 \equiv 18.7$$

Step IV. Check by multiplication, making sure that your result is correct to the nearest third figure: if the product is too small try the next larger number, etc.

27. Find to the nearest third figure the square roots of 12.3; 123; 1098; 10980.

Try Exercise 16, B, page 362. Work to the nearest third figure: if two numbers are equally near to the required root, record them both.

28. The table of squares and square roots. In the table on pages 474, 475 find the square root of 16; of 16.81; 17.64; 18.49. Verify each result. (Notice that the square roots are found in the N column opposite their squares.) Find the square roots of 25; 2500; 26.01; 2601. Find the square roots of 1.44; 144; 1.69; 169; 16900; 1.96; 196. (Notice that these numbers are found above the black line in the table, while 16, 16.81, etc., were found below that line. Your judgment in regard to the size of a square root will tell you whether to look for it above or below the black line.) Find the square root of 4.452. ($\sqrt{4.452}$ is 2.11, as is indicated by the fact that 4.452 is found in the 2.1 row and in the 1 column. Notice that this is an *approximate* square root.) Verify with the help of the table the following approximate square roots: $\sqrt{6.452} = 2.54$; $\sqrt{64.48} = 8.03$; $\sqrt{47.33} = 6.88$. Find to the nearest third figure the square root of 43.05. Result: 6.56. Notice that 43.05 is not found in the table, but that the square of 6.56 is *nearer* to 43.05 than to 43.16 or to any other square in the table. Explain.

Repeat Exercise 16, B, page 362. Give each square root to the nearest third figure. Use the table on pages 474, 475.

29. Interpolation. The use of tables is one of the important and practical lessons of secondary school mathematics; in many kinds of tables it is possible to *interpolate*, that is, to find numbers which lie between those given in the table. It will be well for you to master interpolation now.

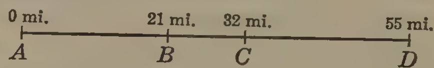
The idea of interpolation is very simple. It is based on the assumption that since, for example, 16.28 is midway between 16.24 and 16.32,

its square root is approximately midway between $\sqrt{16.24}$ and $\sqrt{16.32}$. Hence $\sqrt{16.28} = 4.035$. Verify this by reference to the table.

Again, since 16.25 is $\frac{1}{8}$ of the way from 16.24 to 16.32, its square root is $\frac{1}{8}$ of the way from $\sqrt{16.24}$, or 4.030, to $\sqrt{16.32}$, or 4.040. Since $\frac{1}{8}$ of the 10 "steps" from 4.030 to 4.040 is .8, which is nearer to 1 than to any other integer, therefore $\sqrt{16.25} = 4.031$. Explain.

Find $\sqrt{16.26}$, $\sqrt{16.27}$, $\sqrt{16.28}$, $\sqrt{16.29}$, $\sqrt{16.30}$, $\sqrt{16.31}$.

A study of the accompanying scale and the questions about it will afford a good illustration of interpolation.



Two thirds of the way from B toward C is how many miles from A? (Answer to the nearest tenth of a mile.) Seven eighths of the way from C toward D is how many miles from A? Twenty-five miles from A is what part of the way from B toward C? What part of the way from B toward D? (Give results as decimal fractions to the nearest tenth.)

The form below is one of the forms in which interpolation may be written. As you study these examples, verify or explain each step. Think, which direction? how far?

(*a*) $\sqrt{12.49} = ?$ Estimate 3 +

Step I.

$3.530 = \sqrt{12.46}$ 12.46 is the next number below 12.49 which is found in the table.

$? = \sqrt{12.49}$

$3.540 = \sqrt{12.53}$ 12.53 is the next number above 12.49 which is found in the table.

The square root of 12.49 lies between 3.530 and 3.540. Why?

Step II.

$$\left.\begin{array}{l}\left.\begin{array}{l}3.530 = \sqrt{12.46} \\ 3.53? = \sqrt{12.49}\end{array}\right\}3 \\ 3.540 = \sqrt{12.53}\end{array}\right\}7$$

The tabular difference, that is, the difference between 46 and 53, is 7. (Disregard the decimal point.) From 46 to 49 is 3 of these 7 steps, and we assume that the required square root is $\frac{3}{7}$ of the 10 steps from 3530 to 3540. Now $\frac{3}{7}$ of $10 = \frac{30}{7} = 4$. Hence the required square root is 3.534. Repeat the reasoning until it is well understood.

(*b*) $\sqrt{5907} = ?$ Est. 70 +

$$\left.\begin{array}{l} \left.\begin{array}{l} 76.80 = \sqrt{5898} \\ 76.86 = \sqrt{5907} \end{array}\right\}9 \\ 7690 \ \ = \sqrt{5914} \end{array}\right\}16$$

$\frac{90}{16} = 6$, which is to be added to 76.80.

(*c*) $\sqrt{5.907} = ?$ Est. 2 +

$$\left.\begin{array}{l} \left.\begin{array}{l} 2.330 = \sqrt{5.905} \\ 2.330 = \sqrt{5.907} \end{array}\right\}2 \\ 2.440 = \sqrt{5.954} \end{array}\right\}49$$

$\frac{20}{49} = 0$, to the nearest first figure because 49 into 20 goes nearer to 0 times than to 1 time; or because $\frac{20}{49}$ is less than $\frac{1}{2}$.

(*d*) $\sqrt{634700} = ?$ Est. 800 –

$$\left.\begin{array}{l} \left.\begin{array}{l} 7.960 = \sqrt{63.36} \\ 7.967 = \sqrt{63.47} \end{array}\right\}? \\ 7.970 = \sqrt{63.52} \end{array}\right\}?$$

Explain the fourth figure in the answer.

Repeat Exercise 16, B, page 362. Use the table on pages 474, 475, and extract the roots to the nearest fourth figure.

Solve by formula; extract the roots to the nearest fourth figure:

30. $x^2 - 4x - 23 = 0$ **31.** $x^2 - 2x - 17 = 0$

32. Imaginary roots of quadratic equations. Solve by formula:

$$x^2 - 10x + 30 = 0$$

$$x = \frac{10 \pm \sqrt{100 - 120}}{2} = \frac{10 \pm \sqrt{-20}}{2}$$

Minus 20 is a negative number and hence it has no square root among the numbers you are familiar with. Why? Mathematicians do not stop with this fact. Instead (1) they say that the symbol $\sqrt{-20}$ represents an *imaginary number* (the even roots of negative numbers are called imaginary numbers); (2) they extend the idea of number so that it includes imagi-

nary numbers; (3) they assign to these new numbers such meanings as make possible their use in algebra. See page 336. We can now solve *all* quadratic equations in one unknown. Instead of saying that the equation $x^2 - 10x + 30 = 0$ cannot be solved, we say that it has imaginary roots. What are they? The graph of this equation does not touch the x-axis at all. See page 256. For the present omit the checking of imaginary roots.

Solve the following equations by the formula; check all the solutions but the imaginary ones:

33. $4x^2 - 13x + 9 = 0$

34. $4x^2 - 12x + 9 = 0$

35. $4x^2 - 12x + 10 = 0$

36. $3x^2 - 12x + 10 = 0$

37. $x^2 + 4 = 0$ (Notice that $b = 0$)

38. $2x^2 - 7 = 0$

Kinds of Numbers

Ever since men first learned to count, they have found numbers useful and interesting. At first the small integers were the only numbers known, but as methods of writing numbers were invented, more and more integers were recognized and put to use. There is an infinite number of them.

Common fractions were understood before decimal fractions but the present tendency is to use decimal fractions more and common fractions less; the machinist, for instance, who a generation ago measured in eighths, sixteenths, and sixty-fourths of an inch, now measures in tenths, hundredths, and thousandths of an inch. Irrational numbers fill in all the gaps between the rational numbers on the number scale. In practical use of irrational numbers, we write their approximate value to as great a degree of accuracy as is required by the work in hand. Imaginary numbers complete the number system of algebra.

It is interesting to note that since $\sqrt{3} \times \sqrt{3} = 3$, the product of two irrational numbers may be rational; and since $\sqrt{-3} \times \sqrt{-3} = -3$, by the definition of square root, the product of two imaginary numbers may be real.

Classification of Numbers

I. *Real.* 1. Rational: Integers or fractions in which numerator and denominator are integers. Illustrations: 7, $\frac{2}{3}$, .8, -3.2. 2. Irrational: All real numbers that are not rational. Illustrations: $\sqrt{2}$, π. (Irrational numbers cannot be exactly expressed as common or decimal fractions.)

II. *Not Real.* 1. Pure imaginary numbers; any even root of a negative number. Illustrations: $\sqrt{-2}$, $\sqrt[4]{-6}$, $\sqrt[6]{-1}$. 2. Complex numbers; any sum of a real number and a pure imaginary number. Illustrations: $2+\sqrt{-3}$, $5-2\sqrt{-1}$.

Try Exercises 17 and 18, page 363.

Formulas

Formulas are so useful in the sciences and elsewhere that they should be understood by all educated persons. A formula states a fact or a rule for computation. A formula states the relation between the letters involved. Ralph Waldo Emerson speaks of "the craft of mathematical combination which carries a working plan of the heavens and of the earth in a formula." It is important to be able to read and write formulas and to evaluate, that is, to find the value of one letter when the values of the other letters are known; and often it is convenient to know how to transform a formula into some more useful form.

Translation.

1. Translate into words each formula below.

(*a*) $c = 2\pi r$ circle (*b*) $A = \frac{1}{4}\pi d^2$ circle

(*c*) $V = \frac{4}{3}\pi r^3$ sphere (*d*) $i = prt.$ interest

In what unit is each number in *d* to be expressed?

(*e*) $c^2 = a^2 + b^2$ sides of a right triangle.

(*f*) $a = \sqrt{c^2 - b^2}$ sides of a right triangle.

2. Translate into algebraic symbols:
   (*a*) The perimeter of a rectangle is twice the sum of its length and width.
   (*b*) The area of a trapezoid is the altitude times the average length of its two bases.
   (*c*) The diagonal of a square is the product of a side and the square root of two.
   (*d*) The annual income on d dollars at 5% equals five hundredths of the number of dollars.
   (*e*) The area of a square field which can be inclosed by p feet of fence.
   (*f*) The area of a rectangular field of which the perimeter is p, if the ratio of the length to the width must be 3. (That is, the length is 3 times the width.)
   (*g*) The area of a rectangular field of which the perimeter is $x + 35$ and the ratio of length to width is 5.

Evaluation.

3. If $s = \frac{n}{2}(a + l)$, find s when $a = 1$, $l = 40$, and $n = 4$.

Find l when $s = 80$, $n = 5$, and $a = 1$.

4. $H = \dfrac{nd^2s}{10}$. Find H when $n = 4$, $d = 3\frac{1}{4}$, and $s = 6$.

Solution: $H = \dfrac{4(\frac{13}{4})^2 \cdot 6}{10} = \dfrac{12 \times \frac{169}{16}}{5} = \dfrac{3 \times \frac{169}{4} \times 4}{5 \times 4} = \frac{507}{20} = ?$

5. In the formula above find n when $H = 45$, $d = 3\frac{3}{4}$, and $s = 4$.

Transformation of formulas.

6. Why is it sometimes convenient to transform a formula before evaluating? Consider for instance example 5 above and also the second part of example 3.

7. Solve for f the formula

① $c = \frac{5}{9}(f - 32)$

② $9c = 5f - 160$ ① × 9. Complete and check.

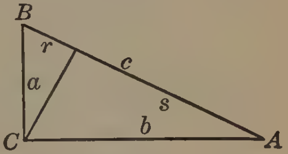
8. Solve for b the formula

① $c^2 = a^2 + b^2$

② $c^2 - a^2 = b^2$ ① $- a^2$

③ $\sqrt{c^2 - a^2} = b$ $\sqrt{②}$

***9.** In the right triangles shown in the figure, $\frac{s}{b} = \frac{b}{c}$ and $\frac{r}{a} = \frac{a}{c}$. From these two equations derive the relation between c, a, and b which is stated in ① of the preceding example. Notice that one of the tasks of algebra is to make such transformations as this for the other sciences.



Try Exercise 19, page 364.

Problem Solution

Introduction. In the solution of verbal problems there is found one of the most important uses of equations and one of the best opportunities to understand and apply the scientific method of thought.

In each problem, numbers and their relations are described in words. To solve a problem we fix our attention on these numbers and represent them and their relations in algebraic symbols. Equations result. Solving the equations is a systematic method of reasoning about the numbers.

1. Give in your own words the substance of the two preceding paragraphs.

2. For success in problem solution, use a systematic plan of attack. Such a plan is outlined on page 169. What are the three habits which you are there advised to form? What are the six "steps" listed there? Study now as much more of the outline as you wish or as your teacher suggests.

3. Contrast the checking of the solution of a verbal problem with the checking of the solution of an equation.

4. If 128 stamps, some 2's and the rest 10's, cost \$3.20, how many stamps of each value are there? Plan of solution:

Step I. List the unknown numbers of the problem.

= no. of 2's	
= " " 10's	
= " " cents for 2's	
= " " " " 10's	
= " " " in all (320)	

When you recognize clearly the numbers involved in a problem, you are well on the way toward its solution.

Take Step I before you take Step II, but do not repeat the list above as is done here. Use abbreviations in writing these verbal descriptions of the numbers involved.

Step II. Represent each number algebraically.

x	= no. of 2's
$128 - x$	= " " 10's
$2x$	= " " cents for 2's
$10(128 - x)$	= " " " " 10's
?	= " " " in all (320)

In taking this step, consider carefully the relations between the numbers. Some of these relations may be stated, others merely implied. As you proceed, have in mind the relation upon which you expect to base Step III.

Step III. *Form an equation.* In this problem we can do so by equating two abbreviations for the same number. (See line 5 of Step II.) Two other plans are suggested on page 171.

$$\text{①}\ 2x + 10(128 - x) = 320$$

Step IV. Solve the equation.

Step V. Check your solution in all the conditions of the problem. Second plan: Make the list of numbers in tabular form.

	Number ×	Value =	Total Value
2's	x	2	$2x$
10's	$128 - x$	10	?
Total	128		320

Explain the table and complete it. Show that it is best to fill in the known numbers first. Show that when you have filled two blanks in one horizontal row, you can fill the third *without reference to the problem*. The use of a table often saves time and aids thinking. It enables you to solve harder problems than you otherwise could. One of the most valuable habits to acquire in the mathematics class is the habit of tabulating data.

Algebraic representation.

5. What is the cost of a articles bought for \$360 a dozen? Of a articles at d dollars a dozen? If the answer is in dollars, change it to cents; if it is in cents, change it to dollars.

6. How many square feet are left uncovered when a 9 ft. by 12 ft. rug is laid on a floor w yards wide and l yards long? Three suggestions:

1. Draw a *diagram* if necessary. 2. *Use a numerical illustration* to help your thinking. 3. Convert all dimensions into the *same units*. Notice that the two members of any equation are to be expressed in units of the same kind.

7. Using the information below, fill the blanks in a similar table of your own:

	Ages 7 years ago	now	in 8 years
B	——	——	——
C	——	——	——

B is now x years old and C is 3 years older.

Try Exercise 20, A, B, page 369, for practice in algebraic representation and in problem analysis. Thorough mastery of this exercise is important.

Solve and check: use linear equations in one unknown. In every problem use the plan of solution described and illustrated above. Take time enough to leave a clear mathematical record.

8. In building a certain type of concrete house, it takes 13 times as long to build the upper part as to build the foundation. If the builder agrees to complete the house in 154 days, how many days can he allow for the foundation?

9. At a baseball game there are three times as many women as children admitted, and eight times as many men as women. The total admissions are 1176. How many are there of each?

10. A jeweler plans to invest twice as much money in jewelry as in watches and 5 times as much in precious stones as in jewelry. In investing $13,910 how much should go into each?

11. A family budget allows twice as much for food as for housing and one half as much for all other expenses as for housing. When the total income is $2800, how much should be allowed for each purpose?

12. If a chair and a table are to be priced so that the table shall cost the buyer $3 more than twice as much as a chair, and if both are to cost him $72, what should be the cost of each?

13. Two supplementary angles have a ratio 8. (This means that one angle is 8 times the other.) How many degrees are there in each?

14. Two supplementary angles have the ratio 19. Find them.

15. One of two complementary angles is 20° more than the other. Find the angles.

16. Find 3 consecutive numbers of which the sum is 84.

17. Find 4 consecutive numbers of which the sum is 130.

18. Eighty stamps, 5's and 10's, cost $6.40; how many stamps are there of each kind?

19. A dealer sold some cars at $920 each and three times as many cheaper cars at $430 each. He received in all $15,470. How many cars of each kind did he sell?

20. How many bills of each kind must I use in order to pay $125 with 5's and 10's, using 4 more 5's than 10's? (Arrange the data in tabular form.)

21. A roll of $5 bills and $2 bills contains $66. There are 12 more 2's than 5's. Find the number of each.

22. $1500 is invested, part at 6% and part at 4%, so that the total income is $78 a year. What is the sum invested at each rate?

23. $2400 is invested, part at 4% and part at 3%, so that the income is $92 a year. What sum is invested at each rate?

24. Separate 132 into two parts so that the smaller divided by the larger shall give $\frac{2}{9}$.

25. The difference between two numbers is 52; their quotient is 3 and remainder 12. What are the numbers?

***26.** One angle is one fifth of another angle, and double the smaller angle is the complement of one fourth the larger. How many degrees are there in each angle?

Use pairs of linear equations.

27. Six horses and 11 cows sell for $1780; at the same prices 13 horses and 5 cows sell for $2350. Find the cost of each.

Combine Step I and Step II thus:

Step II. Step I.
Let h = no. of dollars for one horse
Let c = no. of dollars for one cow.

List only as many numbers as you think necessary before writing the equations. Since no direct relation between the price of a horse and of a cow is indicated, it is convenient to represent these prices by *two* unknowns. It is then necessary to form *two* equations.

Step III.

① $6h + 11c = ?$ On your first reading of each of these problems, try
② $13h + \ ? \ = ?$ to select two relations upon which you will base the two equations. Complete and check.

28. Five pounds of sugar and 2 pounds of coffee cost \$1.25. At the same prices 10 pounds of sugar and 7 pounds of coffee cost \$3.55. Find the cost of a pound of each.

29. Five cans of corn and 4 cans of tomatoes cost \$1.27. At the same prices 3 cans of corn and 6 cans of tomatoes cost \$1.41. Find the cost of each.

30. A pound of tea and 6 pounds of sugar cost \$1.08. Sugar went up 50% and tea went up 10%, then the same quantities cost \$1.38. Find the original price of each. (A tabular arrangement of the list is convenient.)

31. Two pounds of coffee and 3 pounds of butter cost \$2.38. Coffee went up 25%, and butter went up 20%, then the cost of the same order was \$2.90. Find the original price of each.

32. Before the sale 10 handkerchiefs and 4 waists cost \$11.60. By waiting for the sale, when handkerchiefs were marked down 10% and waists 30%, it is possible to make a saving of \$3.08 on these purchases. Find the original prices.

33. Mr. Swanson and Mr. Card are buying a horse for \$550. Mr. Swanson can pay for it if Mr. Card will advance $\frac{1}{3}$ of the money he has in his pocket; and Mr. Card can pay for it if Mr. Swanson will advance $\frac{1}{4}$ of the money in his pocket. How much money has each?

34. Two angles are supplementary and the larger is 45° more than 8 times the smaller. How many degrees are there in each?

35. A man has \$3000 invested, part at 4% and part at 6%. The 4% investment yields \$60 a year more than the 6% investment. What is the amount of each investment?

36. A 7% investment pays \$19 a year more than a 6% investment and the amount of the 7% investment is \$150 more than double the amount of the other investment. Find the amount of each.

Use quadratic equations.

37. A stage driver makes a regular trip of 126 miles. One day he saved 4 hours on the trip by driving 2 miles an hour faster than his usual rate. What was his usual rate?

	D(mi.) =	R(mi. per h.) ×	T (h.)
Regular trip	126	x	?
Faster trip	126	$x+2$	?

Fill the blanks. Do not use the 4 hours given in the problem; save that for the equation.

Explain the equation:

$$\frac{126}{x} = \frac{126}{x+2} + 4$$

Solve and check. Discuss the negative result.

Solve the preceding problem if the numbers are:

	Length of trip	Time saved	Increase in speed
38.	176 mi.	6 h.	3 mi. per h.
39.	120 mi.	4 h.	$1\frac{1}{2}$ mi. per h.

40. A boat which travels 5 miles an hour in still water, goes 21 miles downstream and then 24 miles upstream. The total time is 11 hours. How fast does the stream flow? (Use a tabular arrangement.)

Solve the preceding problem if the numbers in the order given in the problem are:

41.	3 mi. per h.	12 mi.	10 mi.	8 h.
42.	7 mi. per h.	34 mi.	33 mi.	10 h.

43. Forty-six rods of fencing will just inclose a rectangular field of which the diagonal is 17 rods. What are the dimensions of the field?

44. Find the lengths of the sides of a right triangle in which the hypotenuse is 5′ longer and the middle side is 2′ longer than the shortest side.

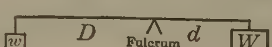
When exact square roots cannot be found, extract the roots to the nearest third or fourth figure.

45. Eighty-four rods of fencing will just inclose a rectangular field of which the diagonal is 30 rods. What are the dimensions of the field?

46. Solve the preceding problem if the diagonal is (*a*) 40 rods; (*b*) 28 rods. (*c*) Try to solve the problem if the diagonal is 5 rods, and comment on the result.

Form equations by substitution in a formula.

47. The fundamental law of all simple machines is briefly expressed as follows: *the effort put in times its distance equals the resistance taken out times its distance.* (Notice, for instance, that when an automobile is in low gear the engine gives great propelling power but each revolution of the engine advances the car only a short distance; in high gear the power is much less but distance per revolution is correspondingly greater.) This law is illustrated simply by the lever. Explain the formula $Dw = Wd$. Notice carefully the significance of the large and of the small letters. An 80-pound weight 5′ from the fulcrum will just balance a 50-pound weight how far from the fulcrum?



Plan of solution:

Step I. Write the required formula.

① $Dw = Wd$

Step II. List the values of the letters.

② $d = 5,\quad w = 80,\quad W = 50,\quad D = ?$

Step III. Substitute the values in the formula.

③ $5 \times 80 = 50\,d$ ② substituted in ①

Step IV. Find the missing value.

Step V. Check in all the conditions of the problem.

48. One hundred pounds 9′ from the fulcrum will balance 30 pounds how far from the fulcrum?

49. A ten-foot stick has a 21-lb. weight suspended from one end and a 49-lb. weight suspended from the other end. At what point will it balance? (Disregard the weight of the stick.) Make a drawing if necessary.

50. Solve the preceding problem for a 21-foot stick, with weights of 18 lb. and 24 lb.

51. The area of the base of a cylinder is 38 square inches. How high must the cylinder be made in order to contain 570 cubic inches? ($V = Bh$)

52. A gallon can is to be made 3″ in diameter. How high must it be made? (1 gal. = 231 cu. in.)

53. What temperature Fahrenheit corresponds (*a*) to 19° centigrade? (*b*) to 60° C? (*c*) to 0° C? (See page 43, problem 7.)

54. A trapezoid containing 90 square feet, is 9′ high, and one base is 7′. How long is the other base?

55. A trapezoid containing 18.08 m^2, is 3.20 meters high and one base is 6.80 meters. How long is the other base?

56. The selling price of merchandise is often fixed so as to give a profit of a certain per cent *of the selling price.* Fix a selling price for hats which cost \$2.94 each, so as to give a profit of 16% of the selling price. *Plan of solution*: The formula is $S - P = C$, or $C + P = S$. Explain.

① $S - .16\,S = 2.94$. Complete and check.

57. Fix a selling price for a piano costing \$218, that will allow a profit of 40% of the selling price.

58. Fix a selling price for radio sets costing \$118, that will give a profit of 15% of the selling price after allowing 28% of the selling price for overhead. Answer to the nearest dollar.

59. Fix a selling price for a radio set costing \$63, that will allow a profit of 22% of the selling price and also allow 28% of the selling price for overhead.

60. In any polygon the relation of the number of diagonals (d) to the number of sides (n) is indicated by the formula $d = \frac{1}{2}n(n-3)$. A polygon with 135 diagonals will have how many sides? A polygon with 77 diagonals will have how many?

61. What sum must be placed at 6% interest for one year, to amount to \$265? ($p + i = a$ and $i = prt$).

62. What sum at 7% interest for one year amounts to \$252.52?

63. What sum at 5% simple interest will amount to \$293.25 in three years?

*64. When Frank was 14 years old his father deposited enough money in the bank to amount to \$2000 for college expenses when Frank was 19. At 4% simple interest what sum did he deposit? Answer to the nearest cent.

*65. What sum must be placed at 6% simple interest to amount to \$3000 in 4 years?

Try Exercise 20, page 369.

Tests of Your Understanding and Skill †

Test A

1. Represent algebraically: (*a*) the difference of the squares of two numbers; (*b*) the cube of the sum of two numbers.

† Read paragraph 5 of *How to Study Algebra*, page xx.

Solve and check:

2. $14 + (2x - 7) = 23 - (3x - 4)$

***3.** $\frac{1}{5}(6x - 1) - \frac{1}{3}(2x - 3) = \frac{x}{10} + 3\frac{2}{3}$

4. $\begin{aligned} 5x + 8y &= 2 \\ 10x - 12y &= 32 \end{aligned}$ **5.** $\frac{x}{a} + 1 = b + c$

Test B

1. Represent algebraically: Ann's age 7 years ago equals one half of her age 5 years hence.

Solve and check:

2. $x^2 + 8x - 5 = x(x - 1) + 5(x + 3)$

3. $\frac{4}{x-2} - \frac{7}{3-x} = \frac{40}{x^2 - 5x + 6}$ **4.** $\begin{aligned} \tfrac{1}{2}A + \tfrac{1}{2}B &= 130 \\ \tfrac{1}{2}A - \tfrac{1}{2}B &= \ \ 70 \end{aligned}$

5. Solve for b, and check: $Z = \frac{1}{2}h(a + b)$

Test C

1. At a minstrel show given by the John Burroughs school there are 360 seats to be sold and it is planned to raise $265 by the sale. How many seats should be sold at $1 and how many at 50 cents?

2. Separate 144 into two parts one of which is $\frac{3}{5}$ of the other.

3. Five pounds of coffee and 4 pounds of tea cost $4.06, and at the same prices, 3 pounds of coffee and 7 pounds of tea cost $4.92. What are the prices?

4. A cone 7′ high and 6′ in diameter will hold how many gallons? Figure $7\frac{1}{2}$ gallons to a cubic foot; use $\frac{22}{7}$ for π. $V = \frac{1}{3}\pi r^2 h$.

5. If $T = \sqrt{s(s-a)(s-b)(s-c)}$ and $s = \frac{1}{2}(a + b + c)$, find T when $a = 5,\ b = 12,\ c = 13$.

Test D

1. (*a*) Multiply $(x-5)(x-6)^2$
   (*b*) Factor $6x^2 - 7x - 20$

2. Solve by factoring: $x^2 - 13x = 140$.

3. Solve by formula: $\dfrac{x-1}{x} - \dfrac{x-1}{6} = 0$

4. Solve and check: $\dfrac{x-5}{x+2} - \dfrac{x+5}{x-3} = \dfrac{1}{2}$. Extract the square root to the nearest fourth figure.

5. The product of two consecutive numbers is 11 more than the sum of the two integers, one next smaller and one next greater than these numbers. What are the numbers?

Test E

1. Answer the thought questions 1–6 on pages 23, 24.

2. What is the "second principle for the use of symbols"?

Part III. Number Relationship or Dependence

How We Use Number Relations

1. What are the three central ideas of algebra as presented in this chapter?

2. Write a formula showing the relation between the cost of a pen (P_1) and the cost of a pencil (P_2) when a pen costs one dollar more than twice as much as a pencil. If a pen and a pencil together cost \$4.75, what is the cost of each?

3. On page 169 you are advised to form what three fundamental habits in solving verbal problems? Discuss the second and third.

4. In the formula $c = 2\pi r$, c and r are so related that any change in the value of one results in a change in the value of the other. Explain and illustrate. (*a*) If r increases, does c increase or decrease? (*b*) If r increases threefold what change in c results? (*c*) If r increases fourfold? (*d*) What do you know about the relative size of c and r? (*e*) The absolute size? (*f*) Since c and r are thought of as changing in value, they are called variables. (*g*) Is π a constant or a variable?

5. Consider the formulas $c = 2\pi r$, $s = \pi r^2$, and $V = \frac{4}{3}\pi r^3$ (circumference and area of a circle and volume of a sphere). It is evident that changes in r have different effects upon c, s, and V. Explain and illustrate. A mathematician would say that *c varies directly as r*; that *s varies as the square of r*; and that *V varies as the cube of r*. Solve the third formula for r and observe that r varies as the cube root of V. Make similar statements concerning s and c. In the formula $a = \frac{1}{b}$, *a varies inversely as b.*

6. Using the words "a varies as" describe the relation between a and b in each formula below. Using the words "b varies as" describe the relation between b and a in each formula below; if necessary transform the formula so that b is expressed in terms of a.

1. $a = 3b$ **2.** $a = 4b^2$ **3.** $a = 6b^5$ **4.** $a = \frac{2}{b}$

5. $a = \frac{3}{b^2}$ **6.** $a = \frac{4}{b^3}$ **7.** $a = 3\sqrt{b}$ **8.** $a = \frac{4}{\sqrt[3]{b}}$

The Relationship $a = rb$, Ratio

1. Ratios are very useful indeed, yet many people do not have a definite idea of what a ratio is. The first fact to learn about a ratio is that a ratio is a *number*, an abstract number, in

fact. The ratio of 12 to 3 is 4; of 4 miles to 8 miles is .5; of 1 pound to 3 pounds is approximately .3; of circumference to diameter of a circle is approximately 3.142; of the weight of lead to the weight of the same volume of water is approximately 11.4; etc. In general, if $a = rb$, and r is a constant, r is the ratio of a to b. (k is often used to represent a constant. The equation then becomes $a = kb$.) In the equation $a = 5\,b$, the ratio of a to b is 5. What is the ratio of the first letter to the second letter in each of the following equations?

$$a = 8\,b; \quad c = \pi d; \quad l = 1.9\,w; \quad x = \tfrac{2}{3}\,y; \quad 5\,x = y; \quad 3\,x = 4\,y.$$

2. Consider the following forms of the equation $a = rb$.

① $a = rb$		This means that the ratio of a to b is the number r by which b is multiplied to obtain a.
② $\dfrac{a}{b} = r$	① $\div\, b$	This means that r, the ratio of a to b, is obtained by dividing a by b.
③ $\dfrac{a}{r} = b$	① $\div\, r$	This means that a divided by the ratio gives b.
④ $\dfrac{1}{r} = \dfrac{b}{a}$	③ $\div\, a$	This means that the ratio of b to a is the reciprocal of the ratio of a to b.

If $a = 4$ and $b = 12$, what is the ratio of a to b? of b to a?
If $a = 10$ and $r = 2$, what is b?
If $b = 15$ and $r = 2$, $a = ?$
Transform the equation $a = 7\,b$ into each of the forms ②, ③, ④, and explain the meaning of each form.

3. When the ratio of two numbers is 1, how do the numbers compare in size?

How to avoid confusion in using ratios. It is evident from the preceding exercise that in using ratios it is not always easy to decide whether to multiply or to divide, or by which number to divide. There are two means of avoiding this confusion: 1. Substituting in the fundamental formula, $a = rb$. 2. Using

judgment in regard to the size of the answer. Apply both of these means in solving the problems on the following pages.

4. The legal length of the United States flag is 1.9 times its width. Express this fact as a formula; $l = ?$ What is the ratio of the length to the width? A flag 2′ high should be how long? A flag 5′ long should be how high?

5. Supply the numbers missing in the table below. In examples 6–10, round off the results to three-figure accuracy, that is, to the nearest third figure. This means three figures in all, not three after the point.

	s Smaller number	l Larger number	$\frac{l}{s}$ Ratio of larger number to smaller	$\frac{s}{l}$ Ratio of smaller number to larger
1.	5	10		
2.	8		3	
3.		16	4	
4.	9			$\frac{1}{7}$
5.		20		$\frac{1}{5}$
6.	22.2	36.3		
*7.		32.7	2.41	
*8.	18.6		5.92	
*9.		56.8		.321
*10.	46.6			.135

Conversion ratios.

6. In most of the countries of the world, distances are measured in kilometers, meters, centimeters, etc.; in the United States and in Great Britain distances are measured in miles, feet, inches, etc.; consequently it is often necessary to convert measurements from one system into the other. For this purpose *conversion ratios* are used. By official definition in the United States 1 meter = 39.370 inches. Rounded off to three-figure accuracy this conversion ratio is 39.4. Explain. If i is the number of inches in a certain distance and m is the number of meters in the same distance, is it correct to write $i = 39.4\,m$ or

$m = 39.4\,i$? (See suggestion 2, page 56.) Using the correct formula, convert to inches to three-figure accuracy: 100 meters; 43.1 meters; .591 *m.*; 3.00 *m.* Express in meters to three-figure accuracy: 51.3″; 29.7″; 43.1″; 8.14″. *In each example ask yourself whether the result should be larger or smaller than the given number.*

7. An athlete in an international contest jumped 6.91 meters. This was .... inches or .... feet. Which of the three units is the largest? Which result will be the largest?

***8.** What is the conversion ratio for converting miles to feet? Using f for the number of feet in a certain distance and m for the number of miles in the same distance, write a formula showing the relation between m and f. Express as miles, to three-figure accuracy: 6370′; 880′. Express as feet: 2.91 mi.; .871 mi.

9. There are 231 cu. in. in a gallon. Write a formula for converting gallons to cubic inches. Make two kinds of problems to be solved by this formula.

Specific gravity.

10. The *specific gravity* of lead is 11.4. This means that the ratio of the weights of equal volumes of lead and water is 11.4, or that lead is .... times as heavy as water. A cubic foot of water weighs 62.5 lb.; what is the weight of a cubic foot of lead?

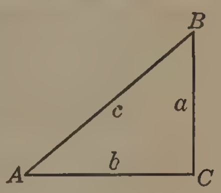
11. A cubic foot of aluminum weighs 168 lb. What is the specific gravity of aluminum?

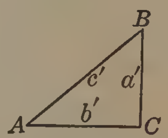
12. Silver weighs 655 pounds per cubic foot. What is its specific gravity?

***13.** Gold weighs 1210 pounds per cubic foot. What is its specific gravity?

14. A stick 4″ by 8″ by 12′ is to be cut from hard wood which has a specific gravity of .813. What will the stick weigh?

15. Similar triangles. In two similar triangles, that is, triangles which have the same shape, the corresponding angles are equal and pairs of corresponding sides have the same ratio. In the similar right triangles shown here the angles marked A are equal, etc., and $\frac{a}{b} = \frac{a'}{b'}$, etc. In a right triangle, for example, in which $A = 40°$, $\frac{a}{b}$ is approximately .8391 no matter how large or how small the triangle may be. In such a triangle if b is 100′ how long is a? If a is 100′ how long is b? (Round off the ratio to the nearest third figure and give the answer to the nearest third figure.





*16. The three sides of a right triangle abc, are 5.00′, 12.0′, and 13.0′ respectively. Find to three-figure accuracy the ratio of a to b; of a to c; of b to c? In a triangle similar to abc, the shortest side is 24.3; find the other two sides.

Other Problems

*17. *The efficiency of a machine* is the ratio of output to input. What is the efficiency of a steam engine which delivered 38,000 foot pounds of useful power when the coal burned gave off 290,000 foot pounds? Answer to the nearest hundredth and then express the answer in per cent.

*18. In raising a 3500 lb. safe 32 ft. (3500 × 32 ft. lb.) with a block and tackle, 170,000 foot pounds of power is used. What is the efficiency of the block and tackle?

19. The radii of two spheres are b and $3\,b$; what is the ratio of their volumes? $\frac{V}{V'} = \frac{\frac{4}{3}\pi b^3}{\frac{4}{3}\pi(?)^3}$. Tell what your answer means.

20. The ratio of two sides of a rectangle is 1.23 and the long side is 29.2′. What is the area? (Round off each computation to the nearest third figure.)

*__21.__ The ratio of two sides of a rectangle is 3.07 and the short side is 7.53′. What is the area?

*__22.__ The ratio of two sides of a rectangular building lot is .473 and the short side is 44.8′. What is the area?

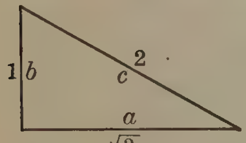
*__23.__ The ratio of two sides of a rectangle is 3.15 and the perimeter is 148′. What are the dimensions?

24. A line b inches long is to be divided into two parts in the ratio 5. How long should each part be?

25. Answer the preceding question if the ratio is m to n. Check the answer.

26. A line 3 c inches long is to be divided into two parts in the ratio 2 a to b. How long should each part be made? Check the answer.

*__27.__ Show that the ratio of a to b in the right triangle illustrated is an irrational number. Give its value to three-figure accuracy. What is the value of $\frac{b}{a}$?



Distance, rate, and time. In solving each problem below, make use of the relationship $d = rt$, and arrange the numbers in tabular form. See page 43, and page 49, example 37.

Solve and check:

1. Allen is to make a trip of 60 miles, walking one half the time at 3 miles an hour and riding a bicycle one half the time at 12 miles an hour. How long will the trip take?

2. A man traveled 340 miles, half the time in an electric car

at 10 miles per hour, and half the time in an automobile at 24 miles per hour. How long did the trip take?

3. In making a 216-mile trip Mr. Walsh expects to travel one third of the time by automobile, averaging 22 miles an hour, and two thirds of the time by boat, averaging 16 miles an hour. How long does he expect the trip to take?

4. The schedule of a river boat calls for 15 miles an hour downstream and 9 miles an hour upstream. Find the rate of the boat in still water and the rate of the stream.

Plan of solution:

	d	$=$	r	$\times$	t
Down	15		$(x + y)$		1
Up	9		$(x - y)$		1

Let x = no. miles per hour in still water. Let y = no. mi. per hour the current flows.

5. A boat went downstream 28 miles in 4 hours and returned in 28 hours. Find the rate of the stream and of the boat.

6. A boat went downstream 36 miles in 3 hours and returned two thirds of the distance in 3 hours. Find the rate of the boat and the rate of the current.

7. A boy coasted down hill at the rate of 20 miles an hour and walked back at the rate of $2\frac{1}{2}$ miles an hour, taking $\frac{1}{2}$ hour for the round trip. How long is the coast?

Try Exercise 21, page 373.

Tests: Dependence

Test A

The formula for the volume of a cone is $C = \frac{1}{3}\pi r^2 h$.

1. As the cone changes in size, which of these letters represent variables?

2. What are the constants in this formula?

3. C varies as .... (Use r)

4. C varies as .... (Use h)

5. Solve the formula for r.

6. r varies as .... (Use C) and as .... (Use h)

7. If $r = 3$ and $h = 1$, express C in terms of π.

8. What is the ratio of C to h?

9. What is the ratio of C to r^2h?

10. The answer to question 9 is what kind of a number?

Test B

1. How far can I ride on a bicycle at 9 miles an hour if I must walk back at $3\frac{1}{2}$ miles an hour and take just $6\frac{1}{4}$ hours for the round trip?

2. A steamboat which goes 11 miles an hour in still water, makes a trip downstream in 4 hours but takes 7 hours to return the same distance. What is the rate of the stream?

3. Mr. Thompson drove 300 miles in 11 hours, driving part of the time at 30 miles an hour, and the rest of the time at 25 miles an hour. How many miles did he drive at each rate?

4. Two brothers start from home and ride on their bicycles in opposite directions, one making 4 miles an hour less than the other. After 5 hours they are 120 miles apart. Find the distance each has ridden.

Part IV. Radicals and Parentheses

Introductory Questions

1. What are the three fundamental notions of algebra as presented in this chapter?

2. What are the two principles for the use of symbols which are stated in this chapter?

3. We have already found some uses for radicals, for example, in formulas, and we have extracted square roots. Before proceeding to later chapters, we need to know (1) how to transform radicals, and (2) how to compute with radicals themselves instead of with their approximate values. The questions to have in mind are: (*a*) What transformations do we need to make? (*b*) How are these transformations made? (*c*) What operations do we need to perform? (*d*) How are these operations performed?

4. The method of study used below is the method of *investigation.* When you are in doubt as to how to perform an operation with radicals, make simple numerical illustrations and decide for yourself. Tell how this method may prove helpful in case you should at some time forget how to perform an operation with radicals. Tell what you think this method has to do with the scientific method of thought. (The time you require to master pages 63–69 is a good measure of your ability and of your methods of study.)

Radicals

1. The meaning of the root sign. Principal roots. Does the symbol $\sqrt{4}$ mean $+2$ or -2 or both? This question cannot be answered except by agreement. It is agreed that the radical sign shall indicate the *principal* root; that is, when a number has two real roots which differ only in sign, the radical represents the positive one. $\sqrt{4} \equiv 2$; $-\sqrt{4} \equiv -2$; $\pm\sqrt{4} \equiv \pm 2$; $\sqrt{9} \equiv 3$. The expression $\sqrt{a^2}$ is usually taken as a if the value of a is unknown.

2. Give the value of $\sqrt{16}$; $-\sqrt{16}$; $\pm\sqrt{25}$; $\sqrt{x^2}$; $\sqrt[3]{a^3}$; $\sqrt[3]{-a^3}$; $\sqrt{-9}$. (Caution: see page 40.)

3. Why we transform radicals and why we operate with them. How can we find the approximate value of $\frac{1}{\sqrt{3}}$? One method is to find the square root of 3, and then to divide in order to find the value of the given fraction. Another method is to write $\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \equiv \frac{\sqrt{3}}{3} \equiv \frac{1}{3}(1.732) \equiv$ etc. Which method is the easier?

4. How can we find the approximate value of $\sqrt{12}$? One method is to find the square root of 12 by use of a table or by some other process. Another method is to write

$$\sqrt{12} \equiv 2\sqrt{3} \equiv 2 \times 1.732 \equiv \text{etc.}$$

The second method is easier if we know that $\sqrt{3} \equiv 1.732$. (It is a good plan to commit to memory $\sqrt{3} = 1.732$, $\sqrt{2} \equiv 1.414$.)

5. How can we find the value of $\sqrt{2} \times \sqrt{18}$? One method is to find the approximate square roots of 2 and of 18 and to multiply. This gives an approximate result. Another method is to write $\sqrt{2} \times \sqrt{18} \equiv \sqrt{36} \equiv 6$. In this illustration, this gives the exact result.

6. Can you now explain why we transform radicals and why we operate with them?

7. What transformations can be made and what operations can be performed? An investigation. With the help of the nine exercises on the next page, decide for yourself what transformations of radicals can be made and what operations with radicals can be performed. Make the investigations carefully; set down your conclusions; and compare them with those of other members of the class. Knowledge gained in this way is easily remembered, and if it should be forgotten can be rediscovered by repeating the investigation. In the left-hand column find the

identities by getting approximate values or by using the definition of square root or in some other way. In the right-hand column find the identities by substituting square numbers such as 4, 9, 16, 25, etc., in place of the letters.

$\sqrt{3} \times \sqrt{5} \overset{?}{\equiv} \sqrt{15}$	I $\sqrt{a} \times \sqrt{b} \overset{?}{\equiv} \sqrt{ab}$
$\sqrt{3} \times \sqrt{3} \overset{?}{\equiv} 3$	II $\sqrt{a} \times \sqrt{a} \overset{?}{\equiv} a$
$\dfrac{\sqrt{10}}{\sqrt{2}} \overset{?}{\equiv} \sqrt{5}$	III $\dfrac{\sqrt{ab}}{\sqrt{a}} \overset{?}{\equiv} \sqrt{b}$
$\sqrt{12} \overset{?}{\equiv} 2\sqrt{3}$	IV $\sqrt{a^2 b} \overset{?}{\equiv} a\sqrt{b}$
$\sqrt{\frac{2}{3}} \overset{?}{\equiv} \sqrt{\frac{6}{9}} \overset{?}{\equiv} \frac{1}{3}\sqrt{6}$	V $\sqrt{\dfrac{a}{b}} \overset{?}{\equiv} \sqrt{\dfrac{a \times b}{b \times b}} \equiv \dfrac{\sqrt{ab}}{b} \overset{?}{\equiv} \dfrac{1}{b}\sqrt{ab}$
$\dfrac{3}{\sqrt{2}} \overset{?}{\equiv} \dfrac{3\sqrt{2}}{\sqrt{2} \times \sqrt{2}} \overset{?}{\equiv} \frac{3}{2}\sqrt{2}$	VI $\dfrac{a}{\sqrt{b}} \overset{?}{\equiv} \dfrac{a\sqrt{b}}{\sqrt{b} \times \sqrt{b}} \overset{?}{\equiv} \dfrac{a\sqrt{b}}{b} \overset{?}{\equiv} \dfrac{a}{b}\sqrt{b}$
$\sqrt{5} + 2\sqrt{5} \overset{?}{\equiv} 3\sqrt{5}$	VII $\sqrt{a} + 2\sqrt{a} \overset{?}{\equiv} 3\sqrt{a}$
$\sqrt{3} + \sqrt{5} \overset{?}{\equiv} \sqrt{8}$	VIII $\sqrt{a} + \sqrt{c} \overset{?}{\equiv} \sqrt{a + c}$
$\sqrt{4 + 9} \overset{?}{\equiv} 2 + 3$	IX $\sqrt{b^2 + c^2} \overset{?}{\equiv} b + c$

Radicals in "simplest form." For the purposes of this course, results containing radicals are to be expressed in "simplest form"; that is, (1) *no removable factor is to be left under the radical sign* (see IV above); (2) *no fraction is to be left under the radical sign* (see V above); (3) *no radical is to be left in a denominator* (see VI above). Commit to memory this definition of *radical expressions in simplest form,* and apply it.

Caution. In practical computations, where approximate numerical values are required, such "simplifications" are not always advantageous. No less an authority than Professor David Eugene Smith says of the example: "'Express in simplest form $\sqrt{8}$ and $\sqrt{\frac{2}{3}}$'; this is an interesting inheritance from the sixteenth century, before decimal frac-

tions and tables had come into general use. The first is already in as simple radical form as it can be, for practical purposes being simpler than $2\sqrt{2}$. The second is in simplest form for computation when written $\sqrt{0.666}$." (*Teachers College Record*, March, 1923.)

Practice with radicals. With the help of what you have just learned, supply such of the numbers missing below as you can. The Roman numerals refer to the identities on page 65.

8. $\sqrt{3} \times \sqrt{7} \equiv \sqrt{?}$ (I,) **9.** $\sqrt{5} \times \sqrt{5} \equiv ?$ (II)

10. $\sqrt{12} \equiv \sqrt{4 \times 3} \equiv ?\sqrt{3}$ (IV)

11. $\sqrt{a^3} \equiv ?\sqrt{a}$ (IV,)

In making such transformations it is helpful to know the square numbers from 2^2 to 20^2 at least. Check each answer by restoring it to its original form.

12. $\sqrt{12} + \sqrt{75} \equiv ?\sqrt{3} + ?\sqrt{3} \equiv ?\sqrt{3}$ (IV, VII)

13. $\sqrt{3} + \sqrt{5} \equiv \sqrt{?}$ (VIII)

Notice that only similar radicals can be united. Sometimes radicals apparently dissimilar are reducible to similar radicals.

14. $\dfrac{\sqrt{18}}{\sqrt{6}} \equiv \sqrt{3}$ (III)

15. $\sqrt{16 + 8\sqrt{3}} \equiv \sqrt{4(4 + 2\sqrt{3})} \equiv 2\sqrt{4 + ?}$ (IV)

16. $\sqrt{\frac{2}{3}} \equiv \sqrt{\dfrac{2 \times 3}{3 \times 3}} \equiv \sqrt{\frac{6}{9}} \equiv \frac{1}{3}\sqrt{6}$ (V)

17. $\dfrac{3}{\sqrt{5}} \equiv \dfrac{3 \times \sqrt{5}}{\sqrt{5} \times \sqrt{5}} \equiv \dfrac{3\sqrt{?}}{5} \equiv \frac{3}{5}\sqrt{?}$ Explain why this process is called *rationalizing* a denominator. What is the *rationalizing factor* used in this illustration?

18. $\sqrt{x^2 + y^2} \equiv ? + ?$ State a "caution" concerning the square root of a binomial, or the number of terms in the square of a binomial.

Solve and check: When answers contain radicals, express them in "simplest form."

19. In a right triangle in which $a = 1$, $b = 1$, and $c = \sqrt{2}$, what is the ratio of a to c? of b to c?

20. A baseball diamond is 90 feet square; find the length of its diagonal.

21. In a right triangle in which $a = 1$, $c = 2$, and $b = \sqrt{3}$, what is the ratio of a to c? of b to c?

In any 30°, 60° right triangle the sides may be represented by x, $2x$, and $x\sqrt{3}$. See for example the triangle of the preceding example. Supply the missing sides of 30°, 60° triangles in the table below.

	x	$2x$	$x\sqrt{3}$
22.	5		
23.		6	
24.		7	
25.			$3\sqrt{3}$
26.			$\sqrt{12}$
***27.**			8
***28.**			10
***29.**			12
***30.**			18
***31.**			21

32. Make a table of square numbers from 2^2 to 30^2.

***33.** Unite $\sqrt{20} + \sqrt{45} + \sqrt{\frac{1}{5}}$ †

Plan of solution: First, simplify each radical as suggested above. Second, unite similar terms.

Try Exercise 22, A, B, page 377.

† Of this exercise Professor David Eugene Smith says: "This is an inherited type of example which teachers have been trying to get rid of for a generation past. It would be difficult to imagine a real situation that would require it."

Multiplication of radicals. Rationalizing factors. Multiply:

1. $\sqrt{6}\times\sqrt{2}\equiv\sqrt{12}\equiv 2\sqrt{?}$

2.
$$\begin{array}{l} \sqrt{3}+\sqrt{5} \\ \sqrt{6}-\sqrt{3} \\ \hline \sqrt{18}+\sqrt{30}-3-\sqrt{15} \end{array}$$

Can this result be simplified? Explain.

3. $(2-\sqrt{3})^2$ **4.** $(2+\sqrt{3})^2$ **5.** $(2-\sqrt{3})(2+\sqrt{3})$

6. $(2\sqrt{3}+3\sqrt{7})^2$ **7.** $(2\sqrt{3}-3\sqrt{7})^2$

8. $(2\sqrt{3}-3\sqrt{7})(2\sqrt{3}+3\sqrt{7})$ ***9.** $(\sqrt{3}-2\sqrt{5}+3\sqrt{7})^2$

10. In examples 5–8 which products are rational? On what conditions is the product of two of the irrational binomials a rational number? Tell without multiplying which of the following products are rational. Verify your answers by multiplication.
$(3\sqrt{5}-2\sqrt{7})(3\sqrt{5}+3\sqrt{7})$; $(3\sqrt{5}-2\sqrt{7})(3\sqrt{5}+2\sqrt{7})$;
$(3\sqrt{5}-2\sqrt{7})(3\sqrt{5}-2\sqrt{7})$; $(3\sqrt{5}-2\sqrt{7})(4\sqrt{5}+2\sqrt{7})$.
Do your conclusions seem to you to have anything to do with the general identity, $(a+b)(a-b)\equiv a^2-b^2$?

11. Rationalize the denominator of $\dfrac{5}{2-\sqrt{3}}$

Plan of work:

$$\frac{5}{2-\sqrt{3}}\equiv\frac{5(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}\equiv ?$$ Explain and complete.

12. $$\frac{\sqrt{6}-3\sqrt{5}}{\sqrt{6}+3\sqrt{5}}\equiv\frac{(\sqrt{6}-3\sqrt{5})(\sqrt{6}-3\sqrt{5})}{(\sqrt{6}+3\sqrt{5})(\sqrt{6}-3\sqrt{5})}\equiv\frac{?}{?}$$

13. Rationalize the denominator of $\dfrac{8}{5-\sqrt{2}}$; of $\dfrac{a\sqrt{b}-c\sqrt{e}}{a\sqrt{b}+c\sqrt{e}}$

Try **Exercise 22, C, D, E, F,** page 378.

Indicated cube roots. Leave under the radical no factor which is a cube: check each answer by transforming it back to its original form.

1. $\sqrt[3]{8}$ **2.** $\sqrt[3]{27}$ **3.** $\sqrt[3]{64}$ **4.** $\sqrt[3]{125}$ **5.** $\sqrt[3]{-216}$ **6.** $\sqrt[3]{343}$

7. Commit to memory the cubes given in the preceding examples.

8. $\sqrt[3]{16} \equiv \sqrt[3]{8 \times 2} \equiv 2\sqrt[3]{?}$ **9.** $\sqrt[3]{40}$ **10.** $\sqrt[3]{56}$

11. $\sqrt[3]{54}$ **12.** $\sqrt[3]{3a^3}$ **13.** $3\sqrt[3]{a^4}$ **14.** $2\sqrt[3]{128a^4}$

15. $a\sqrt[3]{8x}$ **16.** $\frac{1}{2}\sqrt[3]{27a^2}$ **17.** $\sqrt[3]{16a^4xy^2}$

Leave no fraction under a radical:

18. $\sqrt[3]{\frac{3}{4}} \equiv \sqrt[3]{\frac{6}{8}} \equiv \frac{1}{2}\sqrt[3]{?}$ **19.** $\sqrt[3]{\frac{1}{2}}$ **20.** $\sqrt[3]{\frac{a}{3}}$ **21.** $\sqrt[3]{\frac{a}{b}}$

22. $\sqrt[3]{\frac{a}{2b^2}}$ **23.** $(x+2)\sqrt[3]{\frac{1}{(x+2)^2}}$ **24.** $\frac{a-2b}{3}\sqrt[3]{\frac{27}{(a-2b)^2}}$

Unite if possible:

25. $2\sqrt[3]{3} + \sqrt[3]{24} \equiv 2\sqrt[3]{3} + 2\sqrt[3]{3} \equiv ?$

***26.** $3\sqrt[3]{16} - \sqrt[3]{54}$ ***27.** $\sqrt[3]{a} - \sqrt[3]{8a} + \sqrt[3]{27a^4}$

***28.** $\sqrt[3]{a^3b} + \sqrt[3]{8a^3b^4} - 3\sqrt[3]{64b^4}$ ***29.** $\frac{2}{3}\sqrt[3]{\frac{3}{4}} - \frac{5}{2}\sqrt[3]{\frac{2}{9}} + \sqrt[3]{48}$

***30.** $a\sqrt[3]{2x} + \sqrt[3]{128a^3x} - \frac{3}{a}\sqrt[3]{250a^6x}$

***31.** $\sqrt[3]{54} + \sqrt[3]{2000} - 3\sqrt[3]{250}$ ***32.** $3\sqrt[3]{54} - 2\sqrt[3]{16}$

Rationalize the denominators:

33. $\frac{1}{\sqrt[3]{5}} \equiv \frac{1 \times \sqrt[3]{25}}{\sqrt[3]{5} \times \sqrt[3]{25}} \equiv \frac{\sqrt[3]{25}}{?} \equiv \frac{1}{?}\sqrt[3]{25}$

34. $\frac{1}{\sqrt[3]{3}}$ **35.** $\frac{1}{\sqrt[3]{a}}$ **36.** $\frac{2}{\sqrt[3]{6}}$ **37.** $\frac{3a}{\sqrt[3]{9a}}$

***38.** Practice finding cube roots with the help of the table on pages 476, 477. In order to make sure that you are working correctly, begin with the cube roots of 8, 27, 64, etc.

Try Exercise 22, G, page 382, **H,** page 384.

Parentheses

A parenthesis usually serves to group together two or more terms, as, for example, in telling how they are to be operated upon. The bar in a fraction or in a root sign may serve the purpose of a parenthesis. You may be called upon: (*a*) to use a parenthesis in setting up an algebraic expression; (*b*) to perform upon the terms within a parenthesis the operations indicated; (*c*) to remove an expression from a parenthesis; (*d*) to insert an expression into a parenthesis. Some pupils think that parentheses are in some way connected with "changing signs." It is true that when the terms in a parenthesis are *subtracted*, the law for subtraction is followed, but it is that law and not the parenthesis which has to do with the "change in sign."

1. Indicate with the help of parentheses that one half the sum of two numbers is to be multiplied by the square root of their difference and divided by three times their difference, and that the difference of the numbers is to be subtracted from the result.

Perform the operations indicated:

2. $3 - [a - (b - a) - a - 2(a - b) - 3]$. Beginners usually succeed best by removing the inner parentheses first.

3. $(3 + 2)\sqrt{5 + 4} \div (3 - 2) - (4 + 5)\sqrt{4 - 3} \div 3 - 2.$

Insert the 4 or -4 into each "parenthesis" following, without changing the value of the expression. Check by restoring the expression to its original form.

4. $4\sqrt{2}$ Before inserting the 4 in the radical, square it; that is, perform the inverse of the operation indicated. Why?

5. $4\sqrt{a+b}$

6. $4(x-6)^2$ Extract the square root of the 4, that is, perform the inverse of the operation indicated. Why?

7. $4+\frac{3a}{a+b}$ Multiply the 4 by $(a+b)$, that is, perform the inverse of the division indicated.

8. $4-\frac{2x}{x-y}$

9. $3x-4(a+b-c)$

10. $3x-4(a+b-c)$

11. $3x-4\sqrt{a+b-c}$

12. $4+\frac{2}{a+b-c}$

13. $x-4+(2x+5)$

14. $x-4-(2x+5)$

15. In the expression $4+\sqrt{2}$, can the 4 be inserted into the radical as the 4 was in example 4? Notice the importance in such exercises of contrasting *terms* and *factors*.

*16. Comment on the inverse operations used in examples 4–15.

Try Exercise 23, page 385.

Tests of Your Understanding and Skill

If you make a high score on each of the following tests, you may omit further study of Chapter I. If you fail on any of the questions, you should study the theory or the exercises indicated.

Test A

1. Give at least three important suggestions on how to study algebra.

2. Answer the introductory questions found on pages 1 and 23.

3. What three historic meanings of the minus sign are mentioned in this chapter?

4. Show how the laws of signs for subtraction and multiplication can be explained by the words *short* and *over* or *shortage* and *surplus*.

5. Classify the kinds of numbers used in algebra.

6. How many terms has the square of a binomial?

7. Distinguish between the symbol $=$ and the symbol $\equiv$.

8. Mention at least five steps in "A plan for problem solution."

9. Contrast (*a*) checking the solution of an equation; (*b*) checking the solution of a verbal problem; (*c*) checking a transformation, by means of numerical substitution.

*10. Make a brief outline of this chapter.

Test B. Common errors

1. Is $\sqrt{x^2+y^2} \equiv x+y$?

2. What is the value of $\sqrt{4}+25$? of $\sqrt{4+25}$? of $\sqrt{4}+\sqrt{25}$?

3. In simplifying $\frac{4+7}{3\times 5}\times\frac{3}{4}$ is it correct to strike out the 4's? the 3's? Explain.

4. Simplify $\frac{9+12}{3}$; $\frac{9\times 12}{3}$; $\frac{9+11}{3}$.

5. Is $\frac{b-a}{c-d}\equiv\frac{a-b}{c-d}$? Is $\frac{x+y}{w-z}\equiv-\frac{y+x}{w-z}$?

6. If i is the number of inches in a certain distance and f the number of feet in the same distance, does $f=12\,i$ or does $i=12\,f$?

7. By agreement or *convention* is $\sqrt{9}\equiv\pm 3$?

Test C

1. What must be added to $3x^3 - 7x^2 + 5x$ to give

$$7x^3 - 8x^2 + 9?$$

2. What is the remainder when $(a+b)x - (c+e)y + 3(f+g)z$ is subtracted from $2(a+b)x - (c+e)y - 4(f+g)z$?

3. Find the value of

$$4 \times \tfrac{1}{2} - 3\sqrt{5+4} - 5 \cdot 6 + 2(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2}).$$

4. Factor $6a^2x - 19ax + 15x$.

5. $(\frac{2}{3}x - \frac{1}{2}y)(\frac{2}{3}x + \frac{1}{2}y) = ?$

6. $\sqrt{9a - 9b} + \sqrt{4a - 4b} - \sqrt{a - b} = ?\sqrt{?}$

7. In the formula $s = c + \dfrac{2}{a - \dfrac{1}{b}}$ does s become larger or smaller when c increases? When a increases? When b increases? Solve the equation for b and answer corresponding questions for changes in b which result from changes in s and c. (If in doubt in regard to the answer to the second question, for example, assign a fixed value to a and then make a table showing values of s which correspond to various values of b.)

Test D. Formulas

1. In the triangle abc, $a = 20$, $b = 21$, and $c = 29$. Find the area, T, using the formula $T = \sqrt{s(s-a)(s-b)(s-c)}$, in which s is the semi-perimeter.

2. The dimensions of a rectangle are l and w; find the area and the diagonal.

3. Find correct to three significant figures the diagonal of a square 80.0 feet on a side.

4. The diagonal of a square is d; find the side.

5. Find the altitude and area of an equilateral triangle of which the side is a.

Test E. Numerical Equations

Solve and check:

1. $\frac{3y+1}{4} - \frac{x+22}{12} = 3 \qquad x - 2y = 1$

2. $\frac{y+4}{y-5} + \frac{y+3}{y+5} = \frac{2y^2 - 5y + 2}{y^2 - 25}$

3. $x^2 - 8x = 3$ (Extract the root to the nearest third figure.)

4. $\frac{x - 2c}{ax - 4a^2} = \frac{2}{c}$

5. On a 200-mile trip, one train ran 7 miles an hour faster than another and took 1 hour and 45 minutes less time. Find the rate of each train. Discard the negative answer or interpret it. (Arrange the list of numbers in tabular form.)

Test F. Literal Equations

Solve for x. Check by numerical substitution or by substitution of the root in the original equation.

1. $x^2 - 3ax = (3a - x)^2$ **2.** $x + a(x + c) = c + ax$

3. $\frac{a}{x^2 - x - 6} = \frac{1}{a(x-3)}$ **4.** $\frac{x}{b^2 - c^2} = \frac{1}{b - c}$

5. $x^2 - (a + c)x = (x - a)(x + c)$

6. Divide a line $5a$ units long into two parts having the ratio $\frac{2b}{7c}$.

Test G. Problem Solution

1. Before solving any of the following problems study them all and (*a*) find the numbers to list; (*b*) represent each number algebraically; (*c*) state the relations upon which you expect to base the equations; (*d*) form the equations.

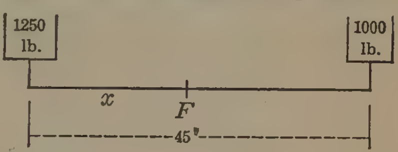
Solve and check: if you fail on any problem, try to decide upon which step you failed.

2. How could a bill of $4.10 be paid using exactly 26 coins, each coin to be either a quarter or a dime?

3. I have $1500 to invest, part in stock paying 6% and the rest in bonds at 4%. How can I divide the money so as to get an annual income of $76?

4. Between Boston and New York, 240 miles, a fast train averages 8 miles an hour more than a slow train and takes 1 hour less time. Find the rates of the trains.

5. One of a pair of horses which are working together is expected to pull 1250 and the other 1000 pounds. When each horse is attached to one end of a 45″ evener, the fulcrum of the evener should be how far from each end?



Test H. Questions Similar to Those Formerly Set by the College Entrance Examination Board

1. Simplify $\sqrt{18} + \sqrt{\frac{1}{2}} + \sqrt{\frac{2}{9}}$

2. Find the numerical value to the nearest hundredth:

$$15\sqrt{\tfrac{7}{3}} - \sqrt{189}$$

3. Rationalize the denominator and find to the nearest hundredth the value of $\dfrac{10 + 3\sqrt{7}}{3 + \sqrt{7}}$

4. Reduce to simplest form, $\frac{1}{2}\sqrt{\frac{3}{4}}+\frac{1}{2}\sqrt{\frac{3}{4}}-7\sqrt{75}$

5. True or false? $\sqrt{\frac{x}{y}} \equiv \frac{\sqrt{xy}}{y}$. Give a reason for your answer.

6. Simplify $\sqrt{\frac{7}{5}} \times \frac{5}{\sqrt{7}}$ *7. Simplify $\frac{\sqrt{4x^2+1}-2x}{\sqrt{4x^2+1}+2x}$

8. Simplify $4a^2-[7a-2-(2a-1)(3-a)]$

9. Solve and verify your result:

$$\tfrac{4}{9}(y+3)=\frac{4y}{9}+\frac{37}{18}-\frac{7y-29}{5y-12}$$

10. Simplify and combine, then find the numerical value to the nearest hundredth, $3\sqrt{\frac{2}{3}}-\sqrt{\frac{3}{2}}+\sqrt{54}$.

11. Solve $a-\frac{9}{2}=\frac{1}{a-1}$.

CHAPTER II

NUMERICAL TRIGONOMETRY

INTRODUCTION

CHAPTER I has supplied the tools and the fundamental ideas of algebra. What are they? Chapter II applies algebra to one of the world's practical problems, indirect measurement; that is, the problem of finding unmeasured angles and distances by means of their relations to known angles and distances. These simple and useful methods were discovered very early in the history of civilization and are still in daily use. Read the *aims* of Chapter II in the Table of Contents.

Questions for Preliminary Discussion

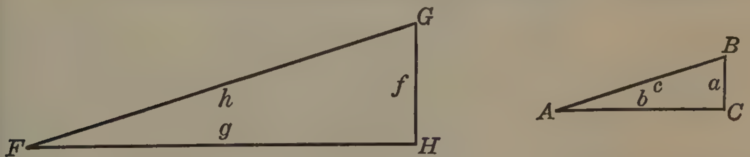
1. What is a right triangle?

2. What is the sum of the two acute angles of any right triangle?

3. What are similar triangles?

4. Two triangles are similar if three angles of one are equal respectively to the three angles of the other. Are two right triangles similar if an acute angle of one is equal to an acute angle of the other?

5. In the similar right triangles ABC and FGH, which angles



are equal? Complete the following equation: $\frac{a}{b} = \frac{f}{?}$. What

other ratios are equal? If the right triangle ABC changes in size without changing shape; that is, if the angles remain unchanged, what ratios will be constant? The lettering illustrated in the smaller triangle is a customary one; notice that C designates the right angle; notice also the relative positions of a and A, b and B, etc.

*6. What is the derivation and meaning of the word trigonometry?

7. What are the steps in the formula method of problem solution?

If you cannot completely answer questions 8–13, keep them in mind as you study the chapter.

*8. What are some of the modern uses of trigonometry? Without indirect measurement could maps of mountainous country be made readily? Could the distance to the moon be known?

9. What is there in the ideas and the symbols of the chapter which makes them so practically useful? (The fact that they are just as useful in the advanced study of mathematics itself as in the applications cannot for obvious reasons be made so apparent in this chapter.)

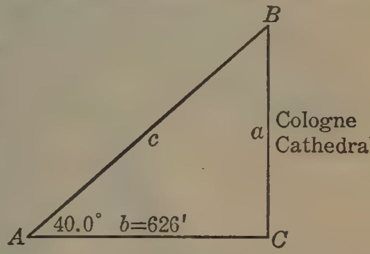
10. In measurement problems why do practical computers round off their results to agree in accuracy with the least accurate measurement?

11. What is the meaning of the expression "three-figure accuracy"? Show that it has nothing to do with the location of the decimal point. Express to three-figure accuracy, that is, to the nearest third figure, each number missing below: 17.3 in. ≡ — ft. ≡ — yd. ≡ — mi. (Notice that zeros at the left end of a number are not counted; zeros at the right are not counted unless there is specific reason for doing so.)

12. Are the numbers in the tables and in the problems of this chapter for the most part approximate or exact?

13. How can your judgment and also scale drawings be used in roughly checking trigonometric computations?

14. An introductory study. In the illustration, a represents the height of a tower of the Cologne cathedral. Find its height, basing your work on the two measurements which have been made and recorded in the figure.



Plan of solution: If we know the ratio of a to b in this triangle, we can find a, which is the required height. This ratio, $\frac{a}{b}$, is constant for all right triangles in which A is 40°,† and is known to be .8391.

① $\frac{a}{b} = .8391$ Given.

② $b = 626$ Given.

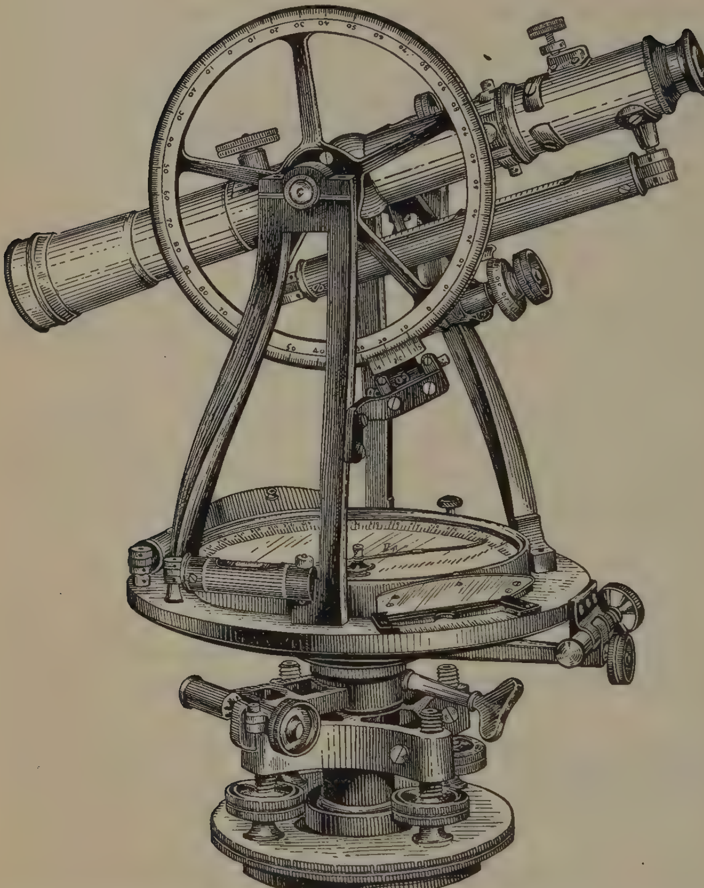
③ $\frac{a}{626} = .839$ ② substituted in ①. The ratio is rounded off to three-figure accuracy in order to agree in accuracy with b.

Complete and check. Give the answer to the nearest third figure. Figures beyond the third will not be reliable.

15. In a right triangle $A = 40°$ and $a = 4.27''$. Find b.

16. In a right triangle $A = 40°$ and $b = 20.8'$. Find a.

† Angles are measured in degrees. One degree is one ninetieth of a right angle. Degrees are subdivided into tenths and hundredths, or into sixtieths called minutes. Minutes are subdivided again into sixtieths called seconds. Pupils unfamiliar with angle measurement should secure protractors and learn to measure angles and to construct angles of given sizes. Protractors will also help to make trigonometry concrete and roughly to check results by means of scale drawings. In field work angles are measured with such instruments as the transit pictured on page 80.

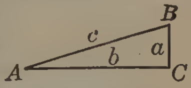


17. In a right triangle $A = 40°$ and $a = 247.8'$. Find b.

Conclusions: Trigonometry measures distances indirectly by means of the ratios of the sides of right triangles. How these ratios are named, and how their numerical values are found, are matters for further study.

The Trigonometric Ratios

1. Names and definitions. In studying an acute angle we often drop a perpendicular from any point in one side to the other side, thereby forming a right triangle. Any such right triangle is called a *triangle of reference* for the given angle. Thus, the right triangle ABC is a triangle of reference for the angle A and also for the angle B.



In a triangle of reference the *tangent of an acute angle is the ratio of the side opposite that angle to the side adjacent:*

$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{a}{b}$$

The *sine of an acute angle is the ratio of the side opposite that angle to the hypotenuse* in the triangle of reference:

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{a}{c}$$

The *cosine of an acute angle is the ratio of the side adjacent to that angle to the hypotenuse* in the triangle of reference:

$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{c}\dagger$$

Commit to memory the words of the three formulas above. They define the ratios which are the working tools of trigonometry. Remember also that a triangle of reference is always a right triangle.

† Three other trigonometric functions are the cotangent (cot), the secant (sec), and the cosecant (csc), which are respectively the reciprocals of the tangent, cosine, and sine. The trigonometric ratios are also called trigonometric *functions.* When a change in the value of one quantity results in a change in the value of another quantity, we may call the one quantity a function of the other. When the angle changes, each of the above defined ratios changes. Hence the term trigonometric *function.*

Referring to the triangle on the preceding page, complete the following statements:

$$\sin B = \frac{?}{?} \equiv \frac{?}{?}, \quad \cos B = \frac{?}{?} \equiv \frac{?}{?},$$

$$\tan B = \frac{?}{?} \equiv \frac{?}{?}, \quad \sin A = \frac{?}{?} \equiv \frac{?}{?}.$$

2. How trigonometric ratios are found. Trigonometric ratios for any acute angle may be found by drawing triangles of reference, measuring their sides, and computing the ratios. Explain and illustrate. Tell how the accuracy of the results obtained in this way depends upon the accuracy of the measurements made.

3. For angles of 30°, 60°, and 45° the trigonometric ratios may be found by the methods suggested on pages 67 and 385. Explain and illustrate. For example, in an isosceles right triangle if the opposite side is 1, the adjacent side will be 1, and the hypotenuse will be $\sqrt{2}$,

therefore $$\sin 45° = \frac{1}{\sqrt{2}} = .7071.$$

4. In a right triangle $a = 15$ and $c = 17$. Find b and then find the sine, cosine, and tangent of A and of B. $(b = \sqrt{c^2 - a^2})$

Try Exercise 24, page 389.

5. Trigonometric ratios of angles from 0° to 90° have been computed by more advanced methods and arranged in tables for convenient reference. With the help of the results of the preceding exercises, and with the aid of your experience in the use of the table of squares, find in the table on pages 488–91,† the sine, cosine, and tangent of angles of 60°, 30°, and 45°. Show that

† The trigonometric tables referred to throughout the text are those in degrees, tenths, and hundredths. This decimal division is to be preferred. However, tables in degrees and minutes are also given. See pages 480–87.

the sin 61° = 0.8746. Find† sin 32.6°; tan 75°; tan 18.3°; cos 38°; cos 87°; cos 51.3°; cos 72.9°. Find cos 22° 18′. ($18' = \frac{18}{60}^\circ$. Represent this common fraction as a decimal fraction to the nearest tenth. The division can be made without pencil.)

***6.** With the help of the table, find which of the trigonometric ratios increase and which decrease as the angle increases in size from 0° to 90°. Compare the cos 10° with the sin 80°; cos 20° with sin 70°, etc. *Cosine* means *complementary sine*, or, *sine of the complement.* Explain.

7. If sin A = .4524, show that A = 26.9°. Or if sin A = .4514, show that A = 26° 50′.

Supply the missing numbers:‡

8. tan ( ) = 0.5938. **9.** cos ( ) = 0.7804.

10. cos ( ) = 0.6807. **11.** sin ( ) = 0.9659.

Find the missing numbers to the nearest tenth of a degree:

12. sin ( ) = 0.2728. **13.** cos ( ) = 0.2540.

14. tan ( ) = 2.819. **15.** cos ( ) = 0.9824.

16. sin ( ) = 0.3245. **17.** cos ( ) = 0.1112.

18. Interpolation. Making the tables do extra work. Given sin 18° and sin 20°, to find sin 19° without reference to the tables. *Plan of work:* Assume that sin 19° is .5 of the way from sin 18° to sin 20°. See also page 37.

$$\left.\begin{array}{l}\sin 18^\circ = 0.3090\\ \sin 19^\circ = \\ \sin 20^\circ = 0.3420\end{array}\right\}\begin{array}{l}\text{Difference, } 330\\ \frac{1}{2} \text{ of } 330 = 165\end{array}$$

If we disregard the decimal point, the tabular difference — that is, the difference between the two numbers obtained from the table — is 330. One half of this difference, 165, added to 3090 suggests that sin 19° = 0.3255. Comparison with the table will show that

† .6° = .6 × 60′ = 36′ ∴ 32.6° = 32° 36′ or 32° 40′ to the nearest 10 minutes. sin 32° 40′ = 0.5398.

‡ If decimal tables are not used, omit exercises 8–17 and practice finding angles of functions read directly from the other tables.

sin 19° = 0.3256. Experiments such as this lead us to the conclusion that small changes in the sine of an angle are approximately but not exactly proportional to changes in the angle. In most parts of the table for changes of less than 1° the kind of interpolation illustrated will give a value that will be correct to the fourth figure.

19. Find sin 18.61°. (To the sine of 18.6° add *one tenth* of the difference between sin 18.6° and sin 18.7°. Arrange the work as in the preceding example.) Or find sin 18° 37′ by adding to the sin 18° 30′ seven tenths of the difference between sin 18° 30′ and sin 18° 40′.

20. Find sin 18.62°, sin 18.63°, and so on to 18.69°.

21. Find tan 5° 33′, tan 5° 34′, tan 81° 44′, tan 63° 3′.

22. Given the cos 66° and the cos 68°, to find cos 67° by interpolation. *Plan of work:* Which way? How far?

$$\left.\begin{array}{l}\cos 66^\circ = 0.4067\\ \cos 67^\circ = \\ \cos 68^\circ = 0.3746\end{array}\right\} \quad 321 \times .5 = 161.$$

To which cosine will you add 161? Or from which will you subtract it? Complete. Check by reference to the table.

23. Cos 20.63° = ? *Plan of work:* Find the number which is .3 of the way from the cos 20.6° toward cos 20.7°.†

$$\left.\begin{array}{l}\cos 20.6^\circ \;\;= 0.9361\\ \cos 20.63^\circ = \\ \cos 20.7^\circ \;\;= 0.9354\end{array}\right\}$$

7 × .3 = 2.1. Complete. Check your result. It should lie between 0.9361 and 0.9354. In your opinion why do beginners make more errors when interpolating for cosines than for sines and tangents?

24. Find cos 38.14°, cos 78.09°, cos 10.13°, cos 15° 17′.

25. Find cos 32.887°. (First round off the angle to the nearest hundredth to match the accuracy of the table.)

26. Find cos 38° 38′. (First express 38′; that is, $\frac{38}{60}°$ as a decimal. With a little practice you can do this without the use of a pencil.)

Try Exercise 25, A, page 390.

† Or cos 20.63′ = cos (20° + .63 × 60′) = cos 20° 38′. From cos 20° 30′ subtract .8 of the difference between cos 20° 30′ and cos 20° 40′. Result, 9359.

27. If $\tan A = 1.931$, find A.

Plan of work:[†]

$$\left.\begin{array}{l}\left.\begin{array}{l}\tan 62.6^\circ = 1.929\\ \tan\quad ?\quad = 1.931\end{array}\right\}2\\ \tan 62.7^\circ = 1.937\end{array}\right\}8$$

From the table. Since our tangent is 2 of the 8 "steps" on the way from 1.929 to 1.937, our angle is $62.6\frac{2}{8}$, or better 66.63°, to the nearest hundredth, which is as far as we can expect it to be reliable. Verify each step of this solution.

Supply the missing numbers:

28. $\sin(\quad) = 0.9477$

29. $\cos(\quad) = 0.8148$

30. $\cos(\quad) = 0.7340$

31. $\cos(\quad) = 0.2222$

Try Exercise 25, B, page 390.

Dealing with Approximate Numbers

All numbers which result from measurement are approximate; so are nearly all the numbers in the tables on pages 474–95, and so are other numbers which have been rounded off; as, .333 for $\frac{1}{3}$ or 3.1416 for π.

Principle. You should clearly understand the following principle: *In computing with approximate numbers, the results can be no more accurate than the least accurate number used.*

Illustration I. Find the area of a rectangle which measures 12.3 ft. by 11.8 ft. These numbers are given to three-figure accuracy.[‡] 12.3 as used here means that the length is nearer to 12.3 ft. than to 12.2 ft. or 12.4 ft. Closer measurement might show that it was 12.25, 12.26, 12.27, 12.28, 12.29, 12.30, 12.31.

† Or follow the same plan using $\tan 62^\circ 30' = 1.921$ and $\tan 62^\circ 40' = 1.935$. $10/14 = .7$. Result $62^\circ 37'$.

‡ In speaking of numbers in this way, zeros at the left are not counted, and zeros at the right are not counted unless they are significant; that is, unless they supply some information about the size of the number other than the location of the decimal point. 12.3, 0.123, and 0.0123 express three-figure accuracy, but 12.30 expresses four-figure accuracy; otherwise the 0 would not be written. It is impossible to tell whether 3740 expresses three-figure or four-figure accuracy unless we are told whether it is a number given to the nearest ten or to the nearest unit.

12.32, 12.33, or 12.34. Give in the same way the meaning of the other dimension of the rectangle. If we take the *smallest* possible values of both the length and width, the area is 12.25×11.75, or 143.9375 sq. ft. when all the figures of the product are retained. If we take the *largest* possible values, the area is 12.34×11.84, or 146.1056 sq. ft. when all figures are retained. Contrast these two results and explain the fact that *the product of two three-figure approximate numbers will not be reliable to more than three figures.* Make a similar statement concerning four-figure approximate numbers.

Illustration II. $\sqrt{5} = 2.24$, and $\sqrt{3} = 1.73$, $\sqrt{5} \times \sqrt{3} \equiv \sqrt{15}$, and $2.24 \times 1.73 = 3.8752$; hence $\sqrt{15}$ apparently should equal 3.8752. Actually $\sqrt{15} = 3.87298$. That is, our product is not reliable beyond the third figure and should, therefore, be rounded off to three-figure accuracy. If we begin with four-figure values, namely, 1.732 and 2.236, the product is 3.872752 which is evidently unreliable beyond the fourth figure and should, therefore, be rounded off to four-figure accuracy.

Illustration III. Find the perimeter of a triangle of which the sides measure 3.0′, 2.77′, and 2.8174′.

$$\begin{array}{r} 3.0??? \\ 2.77?? \\ \underline{2.8174} \end{array}$$

The question marks indicate unknown figures. Can you find the sums of the incomplete columns at the right? Why not? Since this is impossible, we write

$$\begin{array}{r} 3.0 \\ 2.8 \\ \underline{2.8} \\ 8.6 \end{array}$$

and call the perimeter 8.6′.

To find the perimeter to a greater degree of accuracy it would be necessary to *measure* the first side or the first and second sides more accurately.

Practical rules for computation with approximate numbers. *In multiplication and division, round off all numbers to agree in accuracy with the least accurate number (accuracy being measured by the number of significant figures), and round off each re-*

sult to the same number of significant figures:† *in addition and subtraction, round off the numbers so that there are no incomplete columns at the right.*

Practical computers usually carry the work to one extra place and then round off the final result in accordance with the rule stated. For further elementary information about approximate computation and abbreviated methods, see Barber's *Everyday Algebra*, Chapter X. Another abbreviated method of approximate computation is developed in Chapter III of this text. It is called logarithmic computation.

Problems for Trigonometric Solution‡

Introduction. Solve the following problems by the formula method. Draw sketches to help you visualize the situations and select the correct formulas. Use scale drawings to check your results roughly. Plan a careful check of each solution. Arrange your work systematically.

1. Solve the right triangle in which $A = 31.50°$ and $a = 157.3'$.

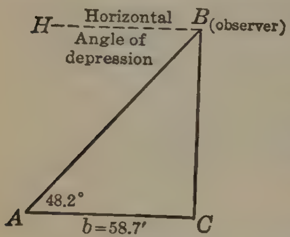
Plan of solution: To *solve* the triangle means to find all missing parts. To find B we use the fact that $A + B + C = 180°$ and that $C = 90°$. To find b we use the formula for the tangent of A. What formula do we use to find c? In the later steps of the solution, why is it better to depend upon given values than upon values which you have computed? Give results to four-figure accuracy. Why? Check by use of the formula for sin B. In problems with approximate data checks are not always exact. Why?

Try Exercise 26, page 391.

† For the purposes of this course this rule may be applied to the measurement of angles as well as of distances. A fuller discussion of the relative accuracy of angles and distances may be postponed.

‡ There are more problems below to be solved by use of natural functions than most classes will need. Problems omitted now may be solved later with the aid of logarithms.

2. The angle of elevation of the top of a tree is 48.2° (48°10′) when measured from a point on the ground 58.7 ft. from the base of the tree. How high is the tree?



Plan of solution: In the sketch, BC represents the tree; A is the angle of *elevation*; HB and AC are horizontal lines. If the observation had been made from B, the angle HBA would be called the angle of depression. An angle of depression or an angle of elevation is an angle between a *horizontal* line and the line of sight. Use tan A.

3. An observer in a balloon 3628 ft. high sighted a signal light at an angle of depression of 18.60° (18° 36′). Find the distance from the balloon to the light and from the light to a point on the ground directly beneath the balloon. Assume that the ground is a horizontal plain.

4. From a point of observation 232.3 ft. above sea level, a buoy is observed at an angle of depression of 43.87° (43° 52′). How far is the buoy from the observer? How far is it from a point at sea level and directly beneath the observer?

5. A certain window in the Charlestown High School building is in the same horizontal plane as the base of Bunker Hill monument and 261 ft. from it. From this window the angle of elevation of the top of the monument is 40.26°. What is the height of the monument?

6. An automobile road under construction is to rise 1.000 ft. for each 14.76 ft. of horizontal distance. At what angle will the road rise?

7. Carpenters speak of the "pitch" of a roof. By pitch they

mean the $\dfrac{\text{height above the eaves}}{\text{the whole width}}$. Find the angle at which a roof rises if it has a one-quarter pitch.

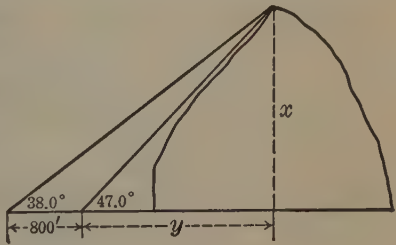
8. Solve the preceding problem if the pitch is two thirds.

9. In laying out a regular ten-sided figure a draftsman drew a circle and divided it into ten equal arcs and drew the chords of these arcs. How many degrees were there in each arc? He wanted each chord to be 2.80″ long. What radius should he use for the circle? (Make a sketch of one of the ten isosceles triangles formed by drawing radii to the ends of the chords.)

10. Five bolt holes are to be bored through a piece of steel at equal intervals, with centers on the circumference of a circle whose radius is 7.30″. What should be the distance between the centers of the holes?

11. A mountain peak 2230 ft. above the level of a horizontal plain casts a shadow extending to a point A on the plain at a time when the sun is at an elevation of 14.7° (14° 40′). How far is it in a straight line from A to the top of the mountain?

***12.** To find the height of a mountain above the plain on which it stands, an observer made the measurements indicated in the drawing. Explain what measurements were made, and find the height of the mountain. In the larger right triangle $\tan 38.0° = \dfrac{x}{800+y}$. Explain. In the smaller right triangle, $\tan 47.0° = \dfrac{x}{y}$. Solve for x.



Round off the result to three-figure accuracy.

***13.** Solve the preceding problem if the angles are 41.0° and 54.0° and the distance 900 ft.

***14.** A house known to be 35 ft. high stands on the top of a hill which rises from a level plain. From a certain point on the plain the angles of elevation of the top and bottom of the house are 36° and 28° respectively. Find the height of the hill.

***15.** Solve the preceding problem if the angles from the same point are 24.0° and 17.0°.

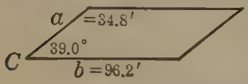
***16.** From a station on the top of a mountain 532 meters high, the angles of depression of two points, both in the same direction from the mountain and on the same horizontal plane, are 9.0° and 18.0°. How far apart are the two points?

***17.** Solve the preceding problem if the angles are 12.0° and 18.0°.

***18. Using a trigonometric formula.** A formula for the area of an oblique triangle is $T = \frac{1}{2}ab \sin C$.† Find the area of an oblique triangle in which side $a = 14.1'$, side $b = 17.2'$, and angle $C = 43.3°$ (43°20′).

***19.** Find the area of an oblique triangle in which $a = 52.6'$, $b = 49.7'$, and $C = 42.0°$.

***20.** Find the area of the parallelogram of this sketch. Observe that the diagonal of a parallelogram divides it into two equal triangles. The area of one triangle is $\frac{1}{2}ab \sin C$. What is the area of the parallelogram?



***21.** Find the area of a parallelogram in which $a = 47.2'$, $b = 63.8'$, and $C = 29.0°$.

***22.** Find the area of a parallelogram in which $a = 92.9$ meters, $b = 54.6$ meters, and $C = 62.0°$.

If further practice is needed, Exercise 26, page 391, may be extended indefinitely, or new constants may be provided for any of the preceding problems. Better still, problems may be

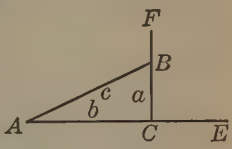
† See example 1, page 128.

based upon practical measurements made by the class with homemade or other instruments. Often a transit discarded by a city surveying department can be had for school use at little or no cost.

If the computations of this chapter are tedious, remember that the following chapter provides a means of shortening them.

Projections

*1. In the right triangle *abc*, b is the *projection* of c on AE and a is the *projection* of c on CF. The projection of a point on a line is the foot of the perpendicular from the point to the line. The projection of a line-segment on another line is the part of the second line between the projections of the end points of the segment. Show that the projection of c on AE is equal to $c \cos A$. $a = c \cos?$ $b = c \sin?$ $a = c \sin?$ $b = c \cos?$



*2. Draw a right triangle, *rst*, with the right angle at T. s is the projection of what line upon what other line? Answer the same question for r. $s = ? \cos?$ $s = ? \sin?$ $r = ? \cos?$ $r = ? \sin?$

Tests

(Read paragraph 5 of "How to Study Algebra," page xx.)

Test A. To Test Your Understanding

1. The hypotenuse of a right triangle is 41′ and another side is 40′. Find without use of tables the sine, cosine, and tangent of the smaller acute angle; that is, the angle opposite the shortest side. Express your results as common fractions.

2. Express 13°18′ in degrees and hundredths. Express 28.14°

in degrees and minutes to the nearest minute. Supply the missing numbers:

$\sin 38°41' =$	$\cos 55.19° =$	$\tan 43°18' =$
$\sin (\quad) = 0.4871$	$\cos (\quad) = 0.6666$	$\tan (\quad) = 1.259$

3. State the practical rule for rounding off when multiplying or dividing approximate numbers; when adding or subtracting approximate numbers.

4. What two principles for the use of symbols are stated in Chapter I? Are they of use in Chapter II? Explain. What three new symbols are introduced in this chapter?

***5.** Solve for $\sin B$: $\dfrac{\sin A}{\sin B} = \dfrac{a}{b}$

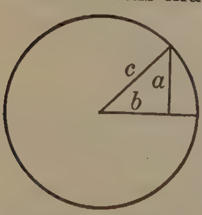
***6.** Solve for $\sin x$: $6 \sin^2 x - \sin x - 1 = 0$

***7.** In the right triangle ABC, $\sin A = ?$ $\cos A = ?$ $\tan A = ?$ Show that $\dfrac{\sin A}{\cos A} = \tan A$.

***8.** Show by numerical illustration that $\sin (A + B) \neq \sin A + \sin B$.

***9.** In the right triangle ABC, show that $a = c \sin A$ and $b = c \cos A$. Using the angle B, give values in this form for a and b.

***10.** In this illustration, if $c = 1$, what line represents the sin of the angle at the center? Discuss changes in the length of this line as the angle at the center increases from 0° to 90°.



***11.** Discuss the value of the cosines of acute angles in the way suggested by the preceding example.

***12.** Discuss as suggested above, the values of the tangents of acute angles. (Draw the triangle so that $b = 1$.)

13. What is an angle of elevation? of depression?

Test B. Applied Trigonometry

1. Solve the right triangle ABC in which $A = 49.9°$ $(49° 50')$ and $c = 327'$.

2. When an airplane is 2200 ft. high, an airdrome is observed at an angle of depression of 18.7°. How far is the airdrome from the airplane? (Answer to the nearest third figure.)

***3.** In an oblique triangle $\dfrac{\sin B}{\sin C} = \dfrac{b}{c}\cdot$ If $b = 29.1'$, $B = 87.7°$, and $C = 59.1°$, find c.

4. The angle of elevation of the top of Washington Monument when measured from a point on the plane of the base and 250.0 ft. away is 65.75° $(65° 45')$. Find the height of the monument.

5. A monument 235 ft. high stands on the bank of a river. The angle of elevation of the top from a point on the opposite bank is 32.55° $(32° 33')$. How wide is the river? (To what degree of accuracy can the answer be determined?)

Test C. Review

Perform the operations indicated:

1. (*a*) $(15 - 3 \times 4)(6 - 8) + 21 \div 7 \times 2$

(*b*) $a^2 - 2\,a^2x \div x + 3\,a^2$

2. Solve and check: $\dfrac{x + 18}{4} - \frac{3}{7}(x - 3) = 4$

3. Solve and check: $7\,x + 6\,y = 5, \qquad 14\,x = 8\,y - 30$

4. $\dfrac{12}{x} - \dfrac{12}{x + 1} = 2$

5. $a = b + \dfrac{c}{x}$ Solve for x.

6. Express as a simple fraction: $\dfrac{\dfrac{x}{3}-3}{2x-\dfrac{5}{2}}$

7. Solve and check: $\dfrac{x-3}{4}-\dfrac{x-5}{5}=\dfrac{x-3}{20}+\dfrac{4}{3x-8}$

8. If $a^2+b^2=c^2$, $a=\sqrt{(c-b)(\ \ ?\ \ )}$.

9. Solve and check if you can. Express any irrational answers to the nearest hundredth:

(*a*) $2x^2+x-15=0$ (*b*) $\dfrac{2-x}{1-x}=3-\dfrac{1-x}{2-x}$

(*c*) $5x^2-17x+100=0$

10. Solve by formula and leave the result in radical form:

$$\frac{5+2x^2}{10}=1+x$$

11. Solve for x and check the result:

$$\frac{cx}{2d}-4d^2=\frac{2dx}{c}-c^2$$

12. Solve for each letter in turn: $T=\pi rl+\pi r^2$

13. If the same number is added to each term of the fraction $\dfrac{c}{f}$ the resulting fraction equals $\frac{1}{5}$? Find the number. Prove by numerical illustration that the resulting formula is correct.

14. Simplify $\sqrt{48}+10\sqrt{\frac{1}{5}}-2\sqrt{45}-\sqrt{\frac{1}{3}}$

15. Find the value to the nearest hundredth of $\dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$.

16. If $x=-2$ and $y=3$, find the value of

$$2(x+y)^2-3(x-1)(y-1)+2xy^2$$

17. $\left(\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)^2 \equiv ?$ **18.** $(2x + 5)(3x - 5) \equiv ?$

19. Simplify each fraction by multiplying its numerator and denominator by a suitable multiplier:

$$\frac{2\frac{2}{3}}{5\frac{1}{2}} \qquad \frac{\frac{a}{b}}{x} \qquad \frac{a}{\frac{b}{x}} \qquad \frac{2 + \frac{1}{3} - \frac{1}{5}}{4\frac{2}{3}} \qquad \frac{a + \frac{a}{b}}{b + \frac{b}{c}}$$

Test D. Problem Solution

1. An investment of \$800, part at $4\frac{1}{2}\%$, and the rest at 5%, pays \$39 a year. How much is invested at each rate?

2. Each of two trains travels 420 miles; one travels 6 miles an hour slower than the other and requires 1 hour and 40 minutes longer to make the trip. Find their rates.

3. One hundred and forty seats are to be sold for a school entertainment, some at 50 cents and the rest at 25 cents. How many must be sold at each price in order to receive \$60?

4. The cost of manufacturing a tire is \$14. At what price must it be sold if 20% of this selling price is to go for overhead and 10% for profit?

5. A woman bought one half lb. of coffee and 3 lb. of butter, for which she paid \$1.65; later she bought 1 lb. of coffee and 2 lb. of butter for \$1.46. Find the cost per pound of butter and of coffee.

CHAPTER III

EXPONENTS AND THEIR USES. LOGARITHMS

PART I. EXPONENTS

CHAPTER I has supplied us with some of the tools and fundamental notions by means of which we can make mathematical studies and investigations, and understand how mathematics is used. Chapter II has shown us how mathematics works on problems of practical measurement and has promised us a method of shortening the somewhat laborious arithmetical calculations involved. Chapter III begins with the theoretical study of exponents, and leads to the understanding of logarithms, which shorten arithmetical calculations.

Preliminary Test

1. What is the meaning of an exponent? Explain and illustrate. Show that your definition holds for positive integral exponents only.

2. x^2 and x^3 are named for what geometric figures? Are the higher powers also named for geometric figures? Explain.

3. Explain and illustrate the general law, $x^a \times x^b = x^{a+b}$. Show that the law depends upon the meaning of an exponent. Show that such a general law is easier to understand, to remember, and to use when stated in symbols than when stated in words.

4. $x^7 \div x^2 = x^?$. Indicate the foregoing division in fractional form and without the use of exponents. State, in the form used in the preceding question, the law of exponents in division. Tell how letters are useful in making *general* statements.

5. Multiply: $x^5 \cdot x^8$; $3x^3 \cdot 3x^3$; $(3x)^3(3x)^4$; $2^3 \cdot 2^4$; $10^3 \cdot 10^5$; $x^l \cdot x^s$; $x^e \cdot x$; $x^{e+1} \cdot x$; $x^{g+1} \cdot x^g$; $x^{g-1} \cdot x$

6. Divide:

$$\frac{x^5}{x^2};\quad \frac{x^3}{x^2};\quad \frac{6x^6}{4x^4};\quad \frac{8x^5}{2x^2};\quad \frac{30a^4b^2}{10ab};\quad \frac{x^a}{x^b};\quad \frac{x^l}{x^r};\quad \frac{x^a}{x};\quad \frac{x^{a+1}}{x};\quad \frac{x^{a+1}}{x^a}$$

7. $(x^2)^3 \equiv xx \cdot xx \cdot xx \equiv x^6$ and $(x^3)^4 \equiv xxx \cdot xxx \cdot xxx \cdot xxx \equiv x^{12}$. State in symbols a *general* law for exponents when raising to a power. Tell why it is important to be able to think out and to express such a general law.

8. Raise to the powers indicated: $(a^5)^3$; $(xy)^2$; $(2ab)^3$; $(\frac{2}{3})^3$; $\left(\frac{x}{y}\right)^2$; $\left(\frac{x^2}{y}\right)^3$; $\left(\frac{5x}{2}\right)^3$; $(x^a)^b$; $(x^b)^a$; $(x^{2s})^n$; $(x^n)^2$; $(x^2)^{3n}$

9. If $(x^2)^3 \equiv x^6$, then $\sqrt[3]{x^6} \equiv x^2$. Explain. What is the *cube root* of a number? Show that $\sqrt[5]{x^{10}} \equiv x^{\frac{10}{5}} \equiv x^2$. Check the result by multiplication. Give numerical illustrations of the general law $\sqrt[r]{x^l} = x^{\frac{l}{r}}$. (Select l and r so that l is a multiple of r.)

10. Extract the roots indicated and check each result:
$\sqrt{x^2}$; $\sqrt[3]{x^3}$; $\sqrt{x^4}$; $\sqrt[3]{x^6}$; $\sqrt[a]{x^a}$; $\sqrt[a]{x^{2a}}$; $\sqrt[r]{x^{3r}}$; $\sqrt[r]{x^l}$; $\sqrt[a]{x^b}$.

11. In what way do exponents make a convenient addition to the symbolism of algebra?

12. What two principles for the use of symbols are stated in Chapter I? How do you expect to apply these principles in your study of exponents?

***13.** To prove the law, $x^l \cdot x^s = x^{l+s}$, write $x^l = x \cdot x \cdot x \cdot \ldots$ to l factors, and $x^s = x \cdot x \cdot x \cdot \ldots$ to s factors, and therefore $x^l \cdot x^s = x \cdot x \cdot x \cdot \ldots$ to ? factors or x^{l+s}. Give a similar proof of the law for exponents in division.

14. What is the value of 3.72×10^3? Of 2.01×10^4?

15. Express in simplest form:

$$ax^3y \times a^2xy^3 \times abxy \qquad \frac{a^2bc^3}{xy^2z^3} \times \frac{x^3y^2z}{ab^2} \div \frac{abcd}{wxy^2x^2}$$

16. *An investigation.* Which of the following statements are identities? How do you know?

(*a*) $x^3 \cdot x^4 \stackrel{?}{\equiv} xxx \cdot xxxx \stackrel{?}{\equiv} x^7$ (*b*) $(xy)^3 \stackrel{?}{\equiv} xy \cdot xy \cdot xy \stackrel{?}{\equiv} x^3y^3$

(*c*) $\left(\frac{x}{y}\right)^4 \stackrel{?}{\equiv} \frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y} \stackrel{?}{\equiv} \frac{x^4}{y^4}$ (*d*) $\frac{x^5}{x^2} \stackrel{?}{\equiv} \frac{xxxxx}{xx} \stackrel{?}{\equiv} x^3$

(*e*) $\frac{6\,a^6}{2\,a^2} \stackrel{?}{\equiv} 3\,a^3$ (*f*) $(2\,x^2)^3 \stackrel{?}{\equiv} 2\,x^2 \cdot 2\,x^2 \cdot 2\,x^2 \stackrel{?}{\equiv} 8\,x^6$

(*g*) $(a^4)^3 \stackrel{?}{\equiv} a^7$ (*h*) $\sqrt[3]{x^{12}} \stackrel{?}{\equiv} x^4$ because $x^4 \cdot x^4 \cdot x^4 \stackrel{?}{\equiv} x^{12}$

(*i*) $\sqrt[4]{x^8} \stackrel{?}{\equiv} x^4$

The Laws of Exponents

1. Study the following four laws of exponents and commit them to memory.

I. $x^l \cdot x^s \equiv x^{l+s}$

II. $\frac{x^l}{x^r} \equiv x^{l-r}$ At least when $l > r$. Other cases will be considered later. See page 104.

III. $(x^l)^s \equiv x^{ls}$

IV. $\sqrt[r]{x^l} \equiv x^{\frac{l}{r}}$ At least when l is a multiple of r.

Assuming that l, s, and r are positive whole numbers, give a numerical illustration of each law and give an explanation to show that you understand the meaning of the law. State each law in your own words. Which two of the laws are the inverses of the other two?

Literal exponents and general statements. Literal exponents are important mainly because they enable us to make such

general statements as those above. From now on you should be able to use them for that purpose.

2. $$\frac{a^2-b^2}{a+b} \equiv a-b; \quad \frac{a^6-b^6}{a^3+b^3} \equiv a^3-b^3$$

In general, $$\frac{a^{2l}-b^{2l}}{a^l+b^l} \equiv a^l-b^l$$

Explain the third identity and prove it by multiplication.

***3.** Give another illustration somewhat similar to the one above.

4. Supply the exponent missing in the following sequence:

1st	2d	3d	4th	...	...	nth
a	ar	ar^2	ar^3			$ar^?$

5. If $l = ar^{n-1}$, then $lr = ar^?$

Exponents in Multiplication and in Long Division

Multiply:

1. $(x^5 - y^5)(x^5 + y^5)$ **2.** $(a^2 - ab + b^2)(a + b)$

3. $(a^2 - 2\,ab + b^2)(a^2 - b^2)$ **4.** $(x^a + x^b)(x^a - x^b)$

5. $(x^a + x^b)^2$ **6.** $(x^2 + xy + y^2)(x^2 - xy + y^2)$

7. Multiply: $$\begin{array}{l} 3\,x^2 - 5\,y^3 \\ \underline{4\,x \;-\; 3\,y} \end{array}$$

Divide: $3\,x^2 - 5\,y^3\overline{)12\,x^3 - 20\,xy^3 - 9\,x^2y + 15\,y^4}$

Show that the division is the inverse of the multiplication.

8. Divide: $15\,a^6 - 17\,a^4b + 14\,a^2b^2$ by $a^3 - 2\,ab$. If there is a remainder, mark it distinctly. Show that both divisor and dividend are arranged according to descending powers of a, and that the dividend is arranged according to ascending powers of b. Do you think that it is generally advisable to use such an arrangement in long division? Give a reason for your answer.

The Degree of a Term. A Lesson in Classification

1. Terms may be classified by means of their exponents. a, x, $3b$, and $2\pi r$ are terms of the first degree; a^2, x^2, $5b^2$, and πr^2 are terms of the second degree; a^3, x^3, $4b^3$, and $\frac{4}{3}\pi r^3$ are terms of the third degree. In general, *the degree of a term is the number of literal factors* (that is, variables) *it contains;* thus the term xy^2z^3 is of the sixth degree in x, y, and z, but of the first degree in x, of the second degree in y, and of the ? degree in z. What is the degree of each of the following terms in each of the variables involved? in all of the variables? x^4; x^2y^2; ax^2; xyz; ab^2c^3; $2\pi rh$; xa^by^c; x^ay.

2. In the following formulas the small letters represent distances. From the degree of each right-hand member (in all the letters involved) decide whether each left-hand member represents a line, an area, or a volume. $V = lwh$. (The right member is a term of the third degree, hence V represents a volume.) $c = 2\pi r$; $p = 4a$; $V = \frac{4}{3}\pi r^3$; $V = \frac{1}{3}\pi r^2h$; $A = \frac{1}{2}bh$; $V = e^3$

3. The degree of an expression. The expression $x^2 + 2x + 3$ is a quadratic expression, or an expression of the second degree. The equation $x^3 - 5x^2 + 7 = 0$ is a cubic equation, or an equation of the third degree. The expression $x^2y^3 - 2xy^2 + xy^4 - 8$ is an expression of the second degree in x, of the fourth degree in y, and of the fifth degree in x and y considered together. Notice that equation $x - 5 + \frac{6}{x} = 0$ is not linear but quadratic. Test it by writing it in integral form. Notice also that $x(x - y)$ is an expression of the second degree in x. In general the degree of an expression or of an equation, in "simplest form," is the degree of the term which contains the greatest number of literal factors. Show that the equation $xy = 8$ is a quadratic equation in x and y, but of the first degree (linear) in x or in y. Show that the equation $x^4 - x^2 - 6 = 0$, although it is in

quadratic form and can be solved by factoring, is an equation of the fourth degree.

What is the degree of each expression or equation below in x? in y? and in x and y?

4. $3x - 1 = y$ **5.** $xy = 15$ **6.** $2\pi x^2 y$

7. $4x^2 - 3y^2 = 1$ **8.** $x(x-1) + y$ **9.** $2\pi x(y + x)$

***10.** $x^5 - x^3 + 3 = 0$ ***11.** $(x-2)(y-3)$ ***12.** $x^2y^2 - xy^4 - x^3$

Try Exercise 27, page 391.

Extending the Meaning of Exponents

The investigation which we are about to make is an illustration of a theoretical study which leads to practical results. We extend the meaning of exponents and as a result we arrive at (1) a convenient substitute for radicals, (2) a convenient form for writing very large or very small numbers, and (3) logarithms, which furnish one of the practical means of shortening all kinds of arithmetic computations except addition and subtraction. You should:

(1) Try to understand how such a theoretical study is made.

(2) Understand thoroughly the symbols and the laws.

(3) Develop skill in the manipulations.

(4) Learn to write numbers in standard form and to use tables of logarithms for the purpose of shortening computations.

1. What is the meaning of x^3? of x^m? Show that your definitions apply to positive integral exponents only, and that such expressions as x^0, $x^{\frac{1}{2}}$, and x^{-2} have no meaning under this definition. As we shall see, it is convenient to make use of fractional, negative, and even zero exponents. In doing so *it would be convenient to have these new exponents follow the four laws of exponents with which we are already familiar. The question is, can we find meanings for these new exponents for which all these four laws will hold?* You should investigate and

try to answer this question for yourself. Since two of the laws of exponents are inverses of the other two, it is sufficient to apply two of the laws to the new exponents; for example I and III.

2. Preliminary investigation. Apply the laws of exponents in the following operations:

$$\frac{x^3}{x^3} \equiv x^? \qquad \frac{x^2}{x^5} \equiv x^? \qquad x^{\frac{1}{2}} \times x^{\frac{1}{2}} \equiv x^? \qquad \sqrt{x} \equiv x^? \qquad \sqrt[5]{x^3} \equiv x^{\frac{?}{?}}$$

Divide a^3 by a; divide the result by a; divide the new result by a. Express each result in terms of a and an exponent. Continue the process until you arrive at a^{-3}. Repeat these steps, expressing the last four results without the use of zero or negative exponents. Form your own ideas about the meanings of the resulting exponents and see how clearly you can state them.

3. Zero as an exponent. Apply the four laws of exponents to x^0 and determine the resulting definition of x^0. By Law I, $x^5 \cdot x^0 \equiv x^5$; but $x^5 \cdot 1 = x^5$, $\therefore x^0 = 1$. By Law II, $x^3 \div x^0 = x^3$, $\therefore x^0 = 1$ (explain); also $\frac{x^4}{x^4} = x^0$, $\therefore x^0 = 1$ (explain). By Law III, $(x^0)^3 = x^0$, but $(1)^3 = 1$, $\therefore x^0 = 1$. By Law IV, $\sqrt[5]{x^0} = x^{\frac{0}{5}} = x^0$, but $\sqrt[5]{1} = 1$, $\therefore x^0 = 1$.

What is the meaning that must be given to x^0 so that it may follow the laws of exponents? In general $\frac{x^n}{x^n} = x^{n-n} = x^0$, but $\frac{x^n}{x^n} = 1$; and $(x^0)^a = x^{a \cdot 0} = x^0$, but $(1)^a = 1$, hence *any number except zero with a zero exponent equals 1.*

4. Find the simplest value of: $5^0 \quad x^0 \quad 3\,a^0 \quad (3\,a)^0 \quad 4 \cdot 7^0 \quad 2^0 x^2 \quad (-10)^0 \quad \frac{a^0}{a^2} \quad \frac{a^2}{a^0} \quad \frac{a^2}{a^2} \quad 10^0 \cdot 10^2 \quad 10^3 \quad 10^2 \quad 10 \quad 10^0$

5. Negative exponents. Apply the laws of exponents to x^{-2} and find the resulting meaning of this symbol. (See Example 3.)

By Law I and because $x^0 \equiv 1$, $x^{-2} \times \frac{x^2}{x^2} \equiv \frac{1}{x^2}$. Explain. This gives what meaning to x^{-2}? By Law II, $\frac{x^5}{x^7} \equiv x^{5-7} \equiv x^{-2}$, but $\frac{x^5}{x^7} \div \frac{x^5}{x^5} = \frac{1}{x^2}$. Does this give the same meaning to x^{-2}?

By Law III $(x^{-3})^2 \equiv x^{-6}$, and $\left(\frac{1}{x^3}\right)^2 = \frac{1}{x^6}$. This gives what meaning to x^{-3} and to x^{-6}?

6. Explain the following transformations:

① $x^a \cdot x^{-a} = 1$ By Law I

② $x^{-a} = \frac{1}{x^a}$ ① $\div\, x^a$

③ $x^a = \frac{1}{x^{-a}}$ ① $\div\, x^{-a}$

7. In general, if the laws of exponents are to hold for negative exponents, then ① $x^a \cdot x^{-a} = 1$

② $x^{-a} = \frac{1}{x^a}$ ① $\div\, x^a$

8. Supply the missing exponents:

a. $x^{-3} \times \frac{x^3}{x^3} \equiv \frac{1}{x^?}$ *b.* $x^{-5} \times \frac{x^5}{x^5} \equiv \frac{1}{x^?}$ *c.* $x^{-7} \equiv \frac{1}{x^?}$

d. $2\,x^{-4} \equiv \frac{2}{x^?}$ *e.* $\frac{2}{x^{-6}} \equiv 2\,x^?$ *f.* $\frac{1}{x^{-3}} \equiv x^?$

g. $\frac{3}{x^{-5}} \equiv 3\,x^?$ *h.* $x^3 \cdot x^2 \equiv \frac{x^3}{x^?}$ *i.* $x^5 \cdot x^3 \equiv \frac{x^5}{x^?}$

Write without negative exponents:

9. 3^{-2} 3^{-3} 4^{-2} 5^{-2} $(-2)^{-2}$ $(-2)^{-3}$ $(-2)^{-4}$

10. $4 \cdot 2^{-4}$ $\quad 5^0 \cdot 4^{-3}$ $\quad (-a)^{-2}$ $\quad (\frac{5}{2})^{-2}$ $\quad (\frac{3}{4})^{-2}$ $\quad 2^0(\frac{2}{3})^{-4}$

11. $\frac{1}{x^{-3}} \cdot \frac{x^3}{x^3} = x^?$ $\quad \frac{2}{3x^{-5}}$ $\quad 10^{-1}$ $\quad 10^{-2}$ $\quad 10^{-3}$ $\quad \frac{1}{10^{-2}}$

12. $\frac{a^{-2}+b^{-3}}{a^{-2}-b^{-3}}$ *Plan of solution:* $\frac{a^{-2}+b^{-3}}{a^{-2}-b^{-3}} \cdot \frac{a^2b^3}{a^2b^3} \equiv \frac{b^3+a^2}{b^3-a^2}$

13. $\frac{x^{-1}+y^{-2}}{x^{-1}y^{-3}}$ **14.** $\frac{1}{a^{-1}b^{-1}}$ **15.** $\frac{1}{a^{-1}+b^{-1}}$ **16.** $\frac{a^{-2}-b^{-1}}{2a^{-3}-b^{-3}}$

Which of the following statements are identities?

17. $\frac{x^2y}{5} \overset{?}{\equiv} \frac{y}{5x^{-2}}$ **18.** $\frac{x^2+y}{5} \overset{?}{\equiv} \frac{y}{5x^{-2}}$ **19.** $\frac{x^2+y}{5} \overset{?}{\equiv} \frac{y}{5+x^{-2}}$

20. $\frac{x^{-2}y}{z} \overset{?}{\equiv} \frac{y}{x^2z}$ **21.** $\frac{2x^2}{y^3} \overset{?}{\equiv} \frac{2y^3}{x^{-2}}$ **22.** $\frac{2a^2}{b^5} \overset{?}{\equiv} \frac{b^{-5}}{2a^{-2}}$

Try **Exercise 28, A,** page 395.

23. Fractional exponents. Investigate the meaning of $x^{\frac{1}{2}}$ by applying several of the laws of exponents.

Suggestions: By Law I, $x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = x$, that is, $x^{\frac{1}{2}}$ is one of the two equal factors of x. Therefore, $x^{\frac{1}{2}} \equiv \sqrt{x}$, by definition of square root.

By Law II, $\quad x \div x^{\frac{1}{2}} = x^{\frac{1}{2}}$, etc.

By Law III, $\quad (x^{\frac{1}{2}})^2 \equiv x^?$, etc.

By Law IV, $\quad \sqrt[2]{x} = x^{\frac{1}{2}}$, etc.

Find the value of:

24. $25^{\frac{1}{2}}$ $\quad (16^3)^{\frac{1}{2}}$

Plan of work: $25^{\frac{1}{2}} \equiv \sqrt{25} \equiv 5$; $(16^3)^{\frac{1}{2}} \equiv \sqrt{16^3} \equiv 4^3 \equiv 64$.

Explain each step. In the second illustration we might have cubed 16 first and then extracted the square root. In your opinion would that have been a shorter procedure?

25. $9^{\frac{1}{2}}$ $\quad 4^{\frac{1}{2}}$ $\quad 36^{\frac{1}{2}}$ $\quad 100^{\frac{1}{2}}$ $\quad (9a^2)^{\frac{1}{2}}$ $\quad (36x^2)^{\frac{1}{2}}$ $\quad (9^3)^{\frac{1}{2}}$

26. $(4^3)^{\frac{1}{2}}$ $\left(\dfrac{4}{9}\right)^{\frac{1}{2}}$ $\left(\dfrac{x^2}{y^4}\right)^{\frac{1}{2}}$ $(9^{\frac{1}{2}})^3$ $(4^{-\frac{1}{2}})^2$

27. $(16^{\frac{1}{2}})^{-3}$ $(25^{\frac{1}{2}})^2$ $(a^{\frac{1}{2}})^2$ $(a^{-\frac{1}{2}})^3$

28. $a^{\frac{1}{2}}b^{-\frac{1}{2}}$ when $a = 49$ and $b = 64$; when $a = 4$ and $b = 100$.

29. If Law I is to hold for fractional exponents, then $x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} \equiv x$; that is, $x^{\frac{1}{3}}$ is one of the three equal factors of x; therefore $x^{\frac{1}{3}} \equiv \sqrt[3]{x}$. See if you can justify this conclusion by the application of Law III; of Law IV.

30. By the method of the preceding example, find the meaning of $x^{\frac{1}{4}}$. What is the value of $16^{\frac{1}{4}}$?

***31.** Show that if Laws III and IV are to hold, $x^{\frac{2}{3}} = \sqrt[3]{x^2}$. What is the value of $125^{\frac{2}{3}}$? of $(\frac{1}{27})^{\frac{2}{3}}$?

From the foregoing considerations it is apparent that the numerator of a fractional exponent indicates a . . ., and that the denominator indicates a . . . . In general, $x^{\frac{l}{r}} \equiv \sqrt[r]{x^l}$.

32. $4^{\frac{3}{2}} \equiv \sqrt{4^3} \equiv 2^3 \equiv ?$ $8^{\frac{2}{3}} \equiv \sqrt[3]{8^2} \equiv 2^2 \equiv ?$

33. Find the simplest value of:

$9^{\frac{3}{2}}$ $27^{\frac{2}{3}}$ $64^{\frac{2}{3}}$ $16^{\frac{3}{4}}$ $4^{.5}$ $9^{1.5}$ $36^{.5}$ $27^{.5}$ $(\sqrt{3} = 1.732)$

34. Express with radical signs: $x^{\frac{3}{4}}$; $a^{\frac{x}{4}}$; $b^{\frac{s}{t}}$.

With the help of the table of square roots find the values of:

35. $10^{.5}$ $10^{.25} \equiv 10^{\frac{1}{4}} \equiv \sqrt[4]{10} \equiv \sqrt{\sqrt{10}} = ?$ $10^{\frac{1}{16}}$

36. $2^{.5}$ $2^{.25}$ $2^{\frac{1}{16}}$

Conclusions:

$$x^0 \equiv 1 \qquad x^{-1} \equiv \frac{1}{x} \qquad x^{-a} \equiv \frac{1}{x^a}$$

$$x^{\frac{1}{2}} \equiv \sqrt{x} \qquad x^{\frac{2}{3}} \equiv \sqrt[3]{x^2} \qquad x^{\frac{l}{r}} = \sqrt[r]{x^l}$$

Try Exercises 28, B and 29, pages 396, 397.

Radicals and Fractional Exponents

We have, now, two methods of indicating that roots are to be taken; namely, radical signs and fractional exponents. It remains to be discovered which method is the more efficient.

***1.** Multiply $\sqrt{a} \times \sqrt[3]{b}$.

Plan of procedure:

$$\sqrt{a} \equiv a^{\frac{1}{2}} \equiv a^{\frac{3}{6}}$$

$$\sqrt[3]{b} \equiv b^{\frac{1}{3}} \equiv b^{\frac{2}{6}}$$

$$a^{\frac{3}{6}} \times b^{\frac{2}{6}} \equiv a^{\frac{3}{6}}b^{\frac{2}{6}} \equiv (a^3b^2)^{\frac{1}{6}} \equiv \sqrt[6]{?}$$

Explain each step and also the general plan of work. Check by setting $a = 4$ and $b = 8$.

Multiply:

***2.** $\sqrt{3} \times \sqrt[3]{4}$ ***3.** $\sqrt{5} \times \sqrt[3]{5}$ ***4.** $\sqrt{8} \times \sqrt[3]{2}$

***5.** $\sqrt{\frac{1}{125}} \times \sqrt[3]{75}$ ***6.** $\sqrt{a} \times \sqrt[4]{a^3}$ ***7.** $\sqrt{x^3} \times \sqrt[3]{x^2}$

***8.** $\sqrt{x} \times \sqrt[3]{y}$ ***9.** $\sqrt[3]{8} \times \sqrt[6]{16}$ ***10.** $\sqrt[3]{32} \times \sqrt{\frac{1}{50}}$

***11.** $\sqrt{\frac{1}{6}} \times \sqrt[3]{\frac{1}{6^2}}$ ***12.** $\sqrt{ab^4} \times \sqrt[4]{a^2b}$ ***13.** $\sqrt[4]{x^{-3}} \times \sqrt{\frac{1}{x^{-5}}}$

Writing Very Large and Very Small Numbers
A Practical Use of Exponents †

1. To avoid copying all the ciphers when computing with very large or very small numbers, mathematicians write 9.28×10^7 instead of 92,800,000 and 2.96×10^{-5} instead of 0.0000296, etc. This is called writing numbers by powers of ten. Justify the exponent in each of these illustrations.

† This section contains important practical information with which all pupils should be familiar, and it also illustrates the most natural method of finding the characteristics of logarithms.

2. Copy the accompanying table and complete it. Verify each exponent. Study the table until you can cover each column and derive it from the other column.

123,000	$\equiv 1.23 \times 10^5$
12,300	$\equiv 1.23 \times 10^4$
	$\equiv 1.23 \times 10^3$
123	$\equiv 1.23 \times 10^?$
12.3	$\equiv 1.23 \times$
1.23	$\equiv 1.23 \times 10^0$
.123	$\equiv 1.23 \times 10^{-1}$
	$\equiv 1.23 \times 10^{-2}$
.00123	$\equiv 1.23 \times$
.000123	$\equiv 1.23 \times 10^{-4}$

3. Justify the following conclusions: In *representing numbers as powers of* 10, to find the exponent count the digits from the space at the right of the head digit to the decimal point. (To the right +, to the left −. The head digit is the first digit not zero.) In *writing the number in full*, to locate the decimal point, start at the space at the right of the head digit and count to the right (+) or to the left (−) as many places as indicated by the exponent of 10. Give two or three illustrations of each process.

Give the numerical value of:

4. 6.23×10^5 $\quad 3.33 \times 10^2$ $\quad 4.86 \times 10^0$ $\quad 4.99 \times 10^{-1}$

5. 5.23×10^{-7} $\quad 6.235 \times 10^{-6}$ $\quad 8.04 \times 10^3$ $\quad 9.10 \times 10^9$

Write by powers of 10:

6. 87,500,000 840,000,000 1,250,000,000

7. 8,997 777,000 0.000732 0.000032

8. 0.00876 0.0888 4.356 4567000

9. Make a table showing the powers of 10 and their values from 10^6 to 10^{-6}.

10. Light travels 186,000 miles a second. How many miles are there in one light-year; that is, in the distance traveled by light in one year?

Plan of solution: $1.86 \times 10^5 (60 \times 60 \times 24)\, 365 \equiv 1.86 \times 10^5 \times 8.64 \times 10^4 \times 3.65 \times 10^2$. We can now estimate the answer

as approximately $2 \times 9 \times 3 \times 10^{11}$, or 54×10^{11}, or 5.4×10^{12}. Continue the multiplication, rounding off each product to three-figure accuracy. Compare your final result with the estimated result.

11. A physiologist studying a problem about the amount of energy used in human life, needed to know how many seconds a man of 65 years has lived. Approximately how many seconds are there in 65 years?

12. A doctor used a liquid containing 4,500,000 bacteria per cm^3. How many bacteria were there in 250 cm^3. of this liquid?

***13.** A scientist estimated that there are about 1500 million persons in the world; that there are approximately 1000 trees to each person; that each tree has approximately one million leaves; and that each leaf has approximately one million breathing pores. Express the approximate number of breathing pores.

***14.** A table of comparative distances gave the distance from the earth to the pole star as 10^{14} km. This is how many kilometers?

***15.** A man has approximately 2.5×10^{13} blood corpuscles. How many blood corpuscles have 250 men?

***16.** Find the number of inches in 100 miles. (Round off each product to three-figure accuracy.)

***17.** Assuming that the path of the earth around the sun is a circle with a radius of 90,000,000 miles, find the distance traveled in one revolution. In one second.

Try Exercise 30, page 399.

Tests

Test A

If possible perform the operations indicated:

1. $a^6 \cdot a^3$ $\quad a^6 + a^3$ $\quad a^6 \div a^3$ $\quad \sqrt[3]{a^6}$ $\quad x^n \cdot x^2$

2. $a^b \cdot a^c \qquad a^d + a^e \qquad a^f \div a^g \qquad \sqrt[r]{a^l} \qquad x^a \cdot x^b \cdot x^c \cdot x$

3. Write without fractional exponents:

$$2\,a^{\frac{1}{2}} \qquad (3\,a^2)^{\frac{1}{3}} \qquad \frac{1}{2 \cdot 2^{\frac{1}{2}}} \qquad 3\,x^{\frac{2}{3}}y^{\frac{5}{3}}$$

4. Write without denominators, using such negative exponents as may be necessary:

$$\frac{w^2x^2}{y^2z^2} \qquad \frac{5}{2\,x - 3\,y} \qquad \frac{4}{3\,x^4} \qquad \frac{3\,x^4}{4}$$

5. Write without root signs and without negative exponents:

$$\sqrt{2\,x} \qquad 3\sqrt{x^3} \qquad \frac{x}{2\sqrt{ab^3}} \qquad 3\sqrt[3]{x^{-2}y^az^{-1}}$$

Test B

Find the numerical values. Give principal roots only.

1. $10^{-4} \quad 81^{\frac{1}{2}} \quad 27^{\frac{2}{3}} \quad (-8)^{-\frac{1}{3}}$
2. $4^0 \cdot 3^{-2} \quad 81^{.75} \quad (15\frac{5}{8})^{-\frac{2}{3}} \quad (\frac{1}{64})^{-\frac{3}{2}}$
3. $8^0 - (\frac{1}{4})^{-\frac{3}{2}} + (3\,x)^0$
4. Find the value of $x^{\frac{1}{2}} - x^0 + 2\,x^{\frac{2}{3}} + x^{-1}$ when $x = \frac{1}{64}$

Test C

Perform the operations indicated:

1. $(x^{\frac{2}{3}} - 4)(x^{\frac{2}{3}} + 5)$
2. $(x^0 - x^{-1})(x^0 + x^{-1})$
3. $(x^{-4} - x^{-2}y + y^2) \div x^{-2}y$
4. $(x^{\frac{3}{2}} - y^{\frac{3}{2}}) \div (x^{\frac{1}{2}} - y^{\frac{1}{2}})$
5. $(x^{n-1} - x^n)(x - x^n)$
6. $(x^{-3} - y^{-3}) \div (x^{-1} - y^{-1})$

Test D

1. Represent by powers of ten 327,000,000; 0.000246.
2. Give the value of 5.930×10^{-6}; 8.92×10^0.
3. Represent by powers of ten the value of $\dfrac{713}{10{,}000{,}000}$.

4. Find to three-figure accuracy the value of $10^{1.5}$.

5. Find to three-figure accuracy the value of $2^2 \cdot 2^{\frac{1}{4}} \cdot \sqrt[4]{2}$.

Test E. Parentheses

Object: To see if you understand how parentheses are used and why. Before beginning the test, make sure that you understand what the *inverse* of an *operation* is and how you are to use inverse operations here.

Supply the missing quantities:

1. $2(4-3\,a) \equiv (?)$ $\quad 2+(4-3\,a) \equiv (?)$ $\quad 2-(4-3\,a) \equiv -\,(?)$

2. $5-3\,a+b \equiv 5-\overline{\quad ? \quad}$ $\quad a^2-b^2+2\,bc-c^2 \equiv a^2-(?)$

3. $3\frac{1}{4} \equiv \frac{?}{4}$ $\quad 3 \equiv \frac{?}{a}$ $\quad 3 \equiv \frac{?}{b-c}$

4. $3+\frac{a}{b} \equiv \frac{?}{b}$ $\quad 4-\frac{c-d}{5} \equiv \frac{?}{5}$ $\quad 3-\frac{a-b}{4} \equiv -\frac{?}{4}$

5. $2\sqrt{a} \equiv \sqrt{?}$ $\quad 3(a)^{\frac{1}{2}} \equiv (?)^{\frac{1}{2}}$ $\quad 2\,a^{\frac{1}{3}} \equiv (?)^{\frac{1}{3}}$ $\quad 4\,a^{\frac{2}{3}} \equiv (?)^{\frac{2}{3}}$

6. $3\,a\sqrt[3]{b^2} \equiv \sqrt[3]{?}$ $\quad a^{\frac{2}{3}}b^{\frac{1}{6}} \equiv \sqrt[6]{?}$ $\quad 2\,a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{6}} \equiv \sqrt[6]{?}$

7. $\sqrt{8} \div (4-\sqrt{2}) = ?$ $\quad \sqrt{8} \div 4 - \sqrt{2} = ?$

8. Find the value of b to three-figure accuracy, when $b=\sqrt{c^2-a^2}$, $c = 13.1$, and $a = 11.1$. Factor the binomial if you think it will save time.

Simplify, by removing from the parentheses all removable factors.

9. $(6\,m-8\,mn)$ $\qquad (-\,14\,a^2bc^3-84\,ab^2c^2)$ $\qquad (\sqrt{5}+2\sqrt{10})$

10. $-\,(3\sqrt{a}-12\,a^{\frac{1}{2}}b^{\frac{1}{2}})$ $\qquad (x^2-2\,xy+y^2+x^3-y^3)$

11. $(p^{\frac{2}{3}}r^{\frac{4}{3}}q^{\frac{7}{3}}-p^{\frac{4}{3}}r^{\frac{2}{3}}q^{\frac{7}{3}}-p^{\frac{7}{3}}r^{\frac{2}{3}}q^{\frac{4}{3}})$

12. $(\sqrt{98}-\sqrt{50})$ $\qquad$ *13. $[8(a+b)^2]^{\frac{1}{2}}$

Write without parentheses:

14. $3\,ab(a - b - 1)$ **15.** $(\sqrt{5} - 3)^2$

***16.** $\left(x - \dfrac{1}{x}\right)\left(x - \dfrac{x}{x+1}\right)$

Test F. Equations

Solve, and check rational roots. Find irrational roots to the nearest hundredth.

1. $3\,x + \frac{1}{2}(-\,x + \frac{5}{6}) = 2\,x(x + \frac{4}{3})$

2. $\frac{1}{6}(x - 6\,y) = \frac{4}{5}(2\,x + 2\,y) - 3 \qquad x - 2\,y = 4$

3. $2\,x^2 - 11\,x + 12 = 0$ **4.** $5\,x^2 - x - 11 = 0$

***5.** $x^{\frac{1}{2}} - 17\,x^{\frac{1}{4}} + 16 = 0$ ***6.** $\sin^2 x - \sin x + \frac{6}{25} = 0$

7. A trophy cup of gold and silver weighs four pounds. In order to find the amount of gold it contains, the cup is weighed in water and found to lose one sixteenth of its weight. How much of the cup is pure gold if, when weighed in water, gold loses one nineteenth and silver one tenth of its weight? (Use a tabular form of arrangement.)

PART II. LOGARITHMS

Introduction

We are ready now to apply our knowledge of exponents to practical computation. For this purpose we use tables of *logarithms*. These are, in fact, tables of exponents, and they are used in accordance with the laws of exponents. They are tables used by practical computers to save time in all kinds of long arithmetic computations except addition and subtraction. You should learn: 1. What logarithms are, and the theory of their use in computation; 2. How to use a table of logarithms; 3. The procedure and arrangement of the work in logarithmic computation.

What Logarithms are and How They are Used

1. Starting with the facts that $2^2 \equiv 4$, $2^3 \equiv 8$, $2^{-1} \equiv \frac{1}{2}$ or .5, and $2^{-6} \equiv \frac{1}{64}$ or .015625, explain the accompanying table of powers of 2 and complete it:

Brief Table of Logarithms to the Base 2

Numbers	Exponents or logarithms
$\frac{1}{64}$ or 0.015625	-6
$\frac{1}{32}$ or 0.03125	-5
$\frac{1}{16}$ or 0.0625	-4
$\frac{1}{8}$ or 0.125	-3
$\frac{1}{4}$ or 0.25	
$\frac{1}{2}$ or 0.5	
1	0
1.414	0.5
2	1
2.828	1.5
4	2
5.656	2.5
8	3
	3.5
16	4
	4.5
32	5
64	6
	7
256	8
	9
1024	10
2048	11
4096	12
8192	13
16384	14
32768	15
65536	16
131072	17
262144	18
524288	19
1048576	20

In the identity $2^4 \equiv 16$, 2 is called the *base*, 4 is called the exponent or the logarithm (*log*), and 16 is called the *number*. The identity may be changed from the exponential form, $2^4 \equiv 16$, to the logarithmic form, $\log_2 16 \equiv 4$. This is read, "The logarithm of 16 to the base 2 is 4." Select from the table other identities and write them in the logarithmic form.

2. With the help of the table, multiply 4 by 64.

Solution:

$$\left.\begin{aligned} 4 &= 2^2 \\ 64 &= 2^6 \end{aligned}\right\} \text{From the table.}$$

$4 \cdot 64 = 2^2 \cdot 2^6 = 2^8$ Law I.

Now $2^8 = 256$ From the table.

Explain each step and check the answer.

3. Multiply 512 by 1024. First plan of solution:

$$512 = 2^?$$
$$1024 =$$
$$512 \cdot 1024 = 2^? \cdot 2^? = \qquad \text{etc.}$$

A more convenient arrangement of logarithmic computation is this:

$$\begin{array}{ll} \log \text{result} = \log 512 + \log 1024 & \text{This is the } \textit{formula.} \\ \log \quad 512 = 9 & \text{Notice that the base is not written.} \\ \log \quad 1024 = \underline{10} & \\ \log \underset{\text{result}}{(\quad)} = 19 & \text{Find in the table the number which corresponds} \\ & \text{to the logarithm 19. Verify the result.} \end{array}$$

Multiply by means of logarithms to the base 2. Arrange your work as in the second part of the preceding example. Refer to the table above.

4. 16×512 **5.** 64×128 **6.** 64×64 **7.** 32×2048

8. 16×8192 **9.** 128×512 **10.** $.0625 \times .25$

11. $.015625 \times 128$ **12.** $.125 \times 1024$ **13.** 1.414×5.656

Check the result in example 13 by multiplying the numbers together, and explain why the check is not exact if all the figures are retained but is exact when the result is rounded off to the nearest third figure.

14. Divide 512 by 64 by means of the table of logarithms to the base 2.

Solution I

$$512 = 2^9$$
$$64 = 2^6$$
$$\frac{512}{64} = \frac{2^9}{2^6} = 2^3 = ?$$

(See Law II, page 98)

Solution II

$$\begin{array}{ll} \log 512 = 9 & \\ \log \quad 64 = \underset{\sim}{6} & \text{Note the use of the wavy line} \\ \log \underset{\text{result}}{(\quad)} = 3 & \text{to indicate subtraction.} \end{array}$$

With the help of the table supply the missing number. Verify the result.

Divide as illustrated in the second solution of the preceding example:

15. $\dfrac{1024}{128}$ **16.** $\dfrac{128}{1024}$ **17.** $\dfrac{32768}{512}$ **18.** $\dfrac{4096}{4096}$

19. $\dfrac{32}{5.656}$ **20.** $\dfrac{256}{8192}$ **21.** $\dfrac{32 \times 1024}{256}$

***22.** $\dfrac{64 \times 4096}{1024 \times 8}$ ***23.** $\dfrac{262144 \times 32}{524288 \times 256}$ ***24.** $\dfrac{11.31 \times 5.656}{2048}$

25. Raise 32 to the fourth power:

Solution I

① $32 = 2^5$

② $32^4 = (2^5)^4$ ① raised to the fourth power.

③ $? = 2^{20}$

Solution II, a better arrangement

$$\begin{array}{lr} \log 32 & = \ 5 \\ & \times \ 4 \\ \hline \log \underset{\text{result}}{(\quad)} & = 20 \end{array}$$

With the help of the table supply the missing number.

26. Find the fifth root of 32768.

Solution I

$32768 = 2^{15}$

$32768^{\frac{1}{5}} = (2^{15})^{\frac{1}{5}} = 2^3 = ?$

Solution II

$$\begin{array}{lr} \log 32768 & = 15 \\ & \div \ 5 \\ \log \underset{\text{result}}{(\quad)} & = \ 3 \end{array}$$

Perform the operations indicated. Use the table of logarithms to the base 2:

27. 32^4 **28.** $.25^3$ **29.** $\sqrt[4]{65536}$ **30.** $\sqrt[3]{.015625}$

***31.** $\dfrac{16^5 \times \sqrt[3]{4096}}{\sqrt[10]{1048576}}$ ***32.** $\dfrac{2.828 \times 8^3 \times \sqrt[7]{16384}}{\sqrt[3]{.0625 \times 8192}}$

Conclusion: The laws of logarithms.†

Law I. $\log MN = \log M + \log N$; that is, *the logarithm of a product is the sum of the logarithms of the factors.* This law follows

† In the statement of each law, the words "to the same base" are omitted for the sake of simplicity.

directly from the first law of exponents. Formal proof will be found on page 130.

Law II. $\log \frac{M}{N} = \log M - \log N$; that is, *the logarithm of a quotient (fraction) is the logarithm of the numerator minus the logarithm of the denominator.* This law follows directly from the second law of exponents.

Law III. $\log N^p = p \log N$; that is, *the logarithm of any power of a number is the logarithm of the number multiplied by the index of the power.* This law follows from the third law of exponents. Formal proof is given on page 130.

Law IV. $\log \sqrt[r]{N} = \frac{1}{r} \log N$; that is, *the logarithm of any root of a number is the logarithm of the number divided by the index of the root.* This law follows directly from Law III when p is a fraction, and is thus a special case of Law III.

Commit these laws to memory, or otherwise make sure that you can write them when you need them.

How to Use a Table of Logarithms

1. In the preceding section we have used a short table of logarithms to the base 2. Tables of *common logarithms*, which are the logarithms used in ordinary computations, have 10 as a base. Ten is a convenient base for logarithms because it is the base of our number system; and its use enables us to determine the approximate size of the logarithms of numbers *by counting the places from the decimal point to the space at the right of the head digit* as we did when we represented numbers by powers of 10. See pages 106–108.

2. Numbers between 1 and 10. In the table on pages 478, 479, we find $\log 3.00 = 0.4771$, $\log 3.10 = 0.4914$, $\log 3.11 = 0.4928$, $\log 3.29 = 0.5172$, $\log 7.76 = 0.8899$, etc. Verify these. Notice

that you find column and row just as you did in using the table of squares.

Supply the missing logarithms:

3. $\log 5.00 = 0.?$ $\log 5.10 = ?$ **5.** $\log 7.77$, $\log 9.9$

4. $\log 5.11$, $\log 5.18$ **6.** $\log 1$, $\log 1.06$

7. Numbers less than 1 or greater than 10. Find $\log 36.0$; $\log 360$; $\log 3600$.

These logarithms are not given in the table, but our knowledge of the laws of logarithms and of exponents will enable us to find them.

$36 = 3.60 \times 10^1$, therefore $\log 36 = \log 3.60 + \log 10$. Why?

$$\begin{aligned} \log 3.60 &= 0.5563, \text{ from table} \\ \log\ 10 &= \underline{1.\qquad} \\ \log 36.0 &= 1.5563, \text{ by Law I.} \end{aligned}$$

In the same way,

$360 = 3.60 \times 10^2$, therefore $\log 360 = \log 3.60 + \log 10^2$

$$\begin{aligned} \log 3.60 &= 0.5563, \text{ from table} \\ \log\ 10^2 &= \underline{2.\qquad} \\ \log\ 360 &= 2.5563, \text{ by Law I.} \end{aligned}$$

Conclusions: (1) The logarithms of most numbers consist of an integer and a fraction. (The integer is called the *characteristic* and the fraction is called the *mantissa.*)

(2) The multiplication of a number by 10 or an integral power of 10 changes the characteristic of its logarithm but not the mantissa.

(3) *The characteristic of the logarithm of a number can be found by counting places from the space at the right of the head digit just as we do in writing a number by a power of* 10.

8. Find the logarithm of 82,500.

Plan I. $82{,}500 = 8.25 \times 10^4$, therefore the characteristic is 4. Find the mantissa; that is, the log 8.25, in the table.

Plan II. *Count from the space at the right of the* 8 *to the decimal point, and set down* 4, etc.

The characteristic can be verified by reference to a table of powers of 10. $10^4 \equiv 10{,}000$ and $10^5 \equiv 100{,}000$. Therefore 82,500, which lies between 10,000 and 100,000, is 10^{4+}.

9. Write the integral part (characteristic) of the logarithm of each number below: Use the method of "conclusion" (3) on page 116. Check by reference to exact powers of 10. 432; 87,900; 478,0005; 940,000; 5.38; 0.483; 0.0376; 0.00423.

10. Find the logarithms of 4.38; 43.8; 438; 438,000; 899; 736,000.

11. Find the logarithm of 0.00376.

By counting from the right of the 3 to the decimal point, or by comparison with exact powers of 10, the characteristic is found to be -3. From the table the mantissa is found to be .5752. Hence $\log 0.00376 = -3 + .5752$. If we perform the subtraction, we get the negative number -2.4238, but a logarithm is not ordinarily so written. Instead, it is written in the form $\overline{3}.5752$† or sometimes $.5752 - 3$. Form the habit of thinking of such a logarithm as a *binomial* in which the characteristic is negative and the mantissa positive. All mantissas in the table are positive.

12. Find the logarithms of 0.00897; 0.0324; 0.473; 5.09.

Try Exercises 31, and 32, A, page 400.

13. Interpolation. Find log 53.42.

† The pupil should know that in many discussions such logarithms are written in the form $7.5762 - 10$. This form is discarded here because there seems to be no reason why the student of algebra should avoid the use of characteristics with negative signs.

Plan of work:

$$\left.\begin{array}{l} \log 53.4 = 1.7275 \\ \log 53.42 = 1.72\,.\,. \\ \log 53.5 = 1.7284 \end{array}\right\} \begin{array}{l} \text{difference} \\ \quad 9 \qquad \times .2 = 1.8, \text{ or } 2. \end{array}$$

Is this 2 to be added to the 75 or to the 84? How do you know? Show that this result is a reasonable one.

Observations: The 1.8 was rounded off to one-figure accuracy because it was unreliable beyond the first place.

The logarithms change continuously as the numbers change. In interpolating we assume that changes in logarithms are proportional to changes in numbers. This is, however, not quite true, and, therefore, interpolation is unreliable when differences are large.

In interpolating *use pencil only when necessary.*

14. Find the logarithm of 5.8127.

First round the number off to four-figure accuracy because figures beyond the fourth will not increase the accuracy of the result. Why?

15. Find the logarithms of 8.247; 324.78; 0.01256; 0.00032579.

Try Exercise 32, B, page 400.

Finding antilogarithms.

16. Multiply 2 by 4 by means of logarithms.

Plan of solution:

$$\begin{array}{ll} \log 2 & = 0.? \\ \log 4 & = \underline{0.?} \\ \log (\quad) & = 0.9031 \end{array}$$

Explain.

Now find 0.9031 in the columns of logarithms in the table (not in the N column). What is the corresponding number; that is, the *antilogarithm* of 0.9031? Is this the product of 2 and 4? Explain.

17. Find the antilogarithm of 3.2601.

Plan of solution: log ( ) = 3.2601.

Step I. Think of the logarithm as 3 + 0.2601. This step is very important because it reminds you that the logarithm is a binomial, only part of which is found in the table.

Step II. Find 0.2601 among the logarithms in the table

(not in the N column) and read the antilogarithm. In this case it is 1.82 because 0.2601 is found in the 1.8 row and in the 2 column.

Step III. Multiply this number by the power of 10 which is indicated by the characteristic; in this illustration, by 10^3, or 1000. In other words, *locate the decimal point by counting three places to the right from the space at the right of the head digit.* The result is 1820.

18. Find the antilogarithm of 2.5752.

19. Supplying the missing numbers:

$\log 8340 = 3.9212$
$\log (\quad) = 1.9212$
$\log (\quad) = 0.9212$
$\log (\quad) = \bar{2}.9212$
$\log (\quad) = \bar{3}.9212$
$\log (\quad) = \bar{6}.9212$

To locate the decimal point in an antilogarithm, start at the space at the right of the head digit and count to the right (+) or to the left (−) as many places as indicated by the characteristic. Compare this rule with the rule for finding the value of a number represented by powers of 10. Verify each result by reference to exact powers of 10. Thus, $10^{3.9210}$ is nearly 10^4, or nearly 10,000, and 8340 is nearly 10,000.

Try Exercise 33, page 401.

Supply the missing numbers. Give each answer to three-figure accuracy. Select in each case the mantissa in the table which is *nearest* to the one you are seeking and give the corresponding number.

20. $\log (\quad) = 0.7405$

21. $\log (\quad) = 1.8989$

22. $\log (\quad) = 2.4881$

23. $\log (\quad) = 6.4130$

24. $\log (\quad) = \bar{1}.9310$

25. $\log (\quad) = \bar{4}.8962$

26. Interpolation in finding antilogarithms. $\log (\quad) = 2.4526$. Give answer to four-figure accuracy.

Plan of work:

$$\left.\begin{array}{l}\left.\begin{array}{l}\log 2.830 = 0.4518\\ \log 2.83? = 0.4526\end{array}\right\}\begin{array}{c}\text{Difference}\\ 8\end{array}\\ \log 2.840 = 0.4533\end{array}\right\}\text{Difference 15 (decimal point disregarded)}$$

The antilog is, therefore, $2.83\frac{8}{15}$ or 2.83? In interpolating use pencil only when necessary. This method of interpolating is usually sufficiently accurate to give one extra figure but not more in the antilogarithm. (Ans. 283.5.)

Supply the missing numbers. Give results to four-figure accuracy.

27. log ( ) = 2.9096 **28.** log ( ) = 1.8363

29. log ( ) = $\bar{3}.0582$ **30.** log ( ) = 0.1777

Try Exercise 34, page 402.

Logarithmic Computation

1. The ordinary cases. Review the laws of logarithms, page 114, and also the arrangement of logarithmic work in the illustrative examples on pages 113–114.

Compute by means of logarithms:

2. 32.84×591.3. *Plan of work:* Estimate $30 \times 600 = 18{,}000$

Formula: log product = log 32.84 + log 591.3. Explain.

log 32.84 =
log 591.3 = ____
log ( ) =
result

Arrange the work systematically *before* referring to the table. It is a good plan to *separate the thinking from the computing.*

3. Multiply 87.0 by 4.32 **4.** Multiply 8.72 by 0.0871 by 64.3

5. Divide 618.5 by 31.47. Estimate $\frac{600}{30} = 20$

Plan of work:

Formula: log quotient = log 618.5 − log 31.47

log 618.5 = 2.
log 31.47 = 1.
log ( ) =
result

6. $73.1 \times 5.78 \div 34.8$ **7.** $6.511 \times 4.891 \div 3.276$

8. -2.632^3 (Negative numbers have no real logarithms. In computing with negative numbers, decide in advance whether the answer is to be positive or negative, and record your decision in the estimate and in the formula. Work as if all the numbers were positive, but give to the answer the correct sign. Contrast the meaning of a negative characteristic with that of a negative number.)

9. $\sqrt[3]{4744}$ (See Solution II, Exercise 26, page 114.)

10. $\sqrt[5]{-347{,}200}$ **11.** $\sqrt{94.38}$

Consider the following rules for successful logarithmic computation:

1. Decide what operations you are to perform and why.
2. Arrange your work systematically.
3. Estimate each result.
4. Check each result.

Read *How to Study Algebra*, paragraph 4, page xx.

The accuracy of the results. In logarithmic computation the degree of accuracy attainable in the results is limited, (1) by the accuracy of the data, and (2) by the accuracy of the tables used. If the data are approximate and given to three-figure accuracy, the results will not be reliable to more than three figures. Continue the discussion, referring if necessary to page 85, Chapter II. Whether the data are exact or approximate, results from the use of four-place tables will not be reliable to more than four significant figures. For ordinary computations this is a sufficient degree of accuracy. When greater accuracy is required, more complete tables must be used. When possible, it is a good plan to extend the computations to one extra place and then to round off the answer.

12. In Exercise 35, page 403, the data are approximate. (Except that numbers given to one or two figures only may be considered exact.) In each example tell to how many significant figures the answer should be given and whether the accuracy

attainable is determined by the data or by the table or by both. Tell also which data should be rounded off before computation is begun.

Try Exercise 35, page 403.

Special devices for special cases. Find by means of logarithms the value of:

13. $\dfrac{0.0237}{3.42}$ Estimate $\dfrac{3).02}{.007}$, or $\frac{2}{100} \times \frac{1}{3} = \frac{2}{300} = .007$

$$\begin{aligned} \log 0.0237 &= \bar{2}.3747 \\ \log \quad 3.42 &= 0.5340 \\ \log\ (\quad) &= \bar{3}.8407 \end{aligned}$$

Notice that in performing this subtraction the 1 "to carry" or "to borrow" when subtracted from the -2 gives -3. Watch these subtractions carefully, and *check them by addition.*

14. $\dfrac{2.358}{0.01243}$ **15.** $\dfrac{0.002769}{0.03548}$ **16.** $\dfrac{0.073459}{0.0076521}$

Operations with logarithms which have negative characteristics are complicated by the fact that these logarithms are binomials, part positive and part negative. The remedy is to *separate these logarithms into their component parts and to deal with them as binomials.* They afford excellent practice in the use of signed numbers.†

Try Exercise 36, page 404.

17. 0.726^5. Estimate, $1^5 = 1$: $.7^5$ is somewhat less than 1.

Formula: log ans. $= 5 \times \log 0.726$

$$\begin{array}{lllll} \log 0.726 = \bar{1}.8609 & \text{or} & 0.8609 - 1 & \text{or} & 9.8609 - 10 \\ \qquad\qquad \times 5 & & \qquad \times 5 & & \qquad \times 5 \\ \hline \log\ (\quad) = ?.3045 & & 4.3045 - 5 & & 49.3045 - 50 \end{array}$$

To find the characteristic, think $5 \times (-1) = -5$ and 4 "to

† The method of augmenting the logarithms so as to avoid some of these difficulties of sign may, of course, be taught if the teacher desires. It is not mentioned here because we have been at some pains to teach the pupil to add and to subtract signed numbers, and it seems better to use than to avoid this opportunity to make practical use of these processes.

carry" makes -1. Explain the second multiplication and show that it gives the same result. Check the multiplication by division. $5\overline{)\bar{1}.3045}$. This binomial is not in good form for division because we cannot carry the division along from the negative term to the positive term. To overcome this difficulty, we rewrite the binomial so that *the head figure is positive and the negative term is divisible by* 5. $5\overline{)4.3045 - 5}$. Compare this with the second multiplication above. Now complete the check.

18. 0.0578^6 **19.** 0.00729^4 **20.** 0.3628^3

21. 0.000321^7

22. $\sqrt[5]{0.0358}$

$$\begin{aligned} \log 0.0358 &= \bar{2}.5539 \\ &\ +5 \qquad -5 \\ &\ 5\overline{)3.5539 - 5} \\ \log\ (\quad) &= 0.7108 - 1 \\ &\ \text{or } \bar{1}.7108 \end{aligned}$$

Explain. Does the second step make the head figure positive and the negative term divisible by 5? ($8.5539 - 10$ may similarly be rewritten $48.5539 - 40$ before dividing by 5.)

Complete and check.

23. $\sqrt[3]{0.276}$ **24.** $\sqrt[4]{0.3894}$ **25.** $\sqrt[5]{0.0237}$ **26.** $\sqrt[6]{0.05248}$

Try Exercise 37, page 405.

27. $\sqrt[3]{\dfrac{0.0176 \times (-342)}{0.238}}$

$$\text{Estimate} \quad -\sqrt[3]{\frac{0.02 \times 300}{\frac{1}{4}}} \equiv -\sqrt[3]{\frac{6}{\frac{1}{4}}} \equiv -\sqrt[3]{24} \equiv -3$$

Formula: $\log\,(-\text{ans.}) = \frac{1}{3}(\log 0.0176 + \log 0.342 - \log 0.238)$

log 0.0176	$= \bar{2}.$	or 8.	-10
log 342	$= 2.$	2.	
log numerator =			
log 0.238	$= \bar{1}.$	9.	-10
	$3\overline{)\quad\quad}$		
log ( ) result	=		

28. $\dfrac{2.963^3 \times \sqrt[5]{30.34}}{-0.9874}$ Estimate $\dfrac{25 \times 2}{-1} = -50$

Formula: $\log \underset{\text{result}}{(-)} = 3(\log 2.963) + \frac{1}{5}(\log 30.34) - \log 0.9874$

$$\begin{array}{lll}
\log 2.963 = 0. & \times 3 = & \\
\log 30.34 = 1. & \times \frac{1}{5} = & \underline{\qquad\qquad} \\
\log \text{numerator} & = & 1.7118 \text{ (verify)} \\
\log 0.9874 & = & \\
\log \underset{\text{result}}{(-\quad)} & & 1.7173 \text{ (verify)}
\end{array}$$

29. $\dfrac{4.236^2 \times \sqrt{82.56}}{18.24^2}$ **30.** $\sqrt{\dfrac{56.43 \times (-3.561)}{0.4273}}$

***31.** $4326(1.035)^6$ **32.** $4800(1.06)^{20}$

***33.** Find A if $\cos A = \sqrt{\dfrac{44.60 \times 0.07120}{17.20}}$

Try Exercise 38, page 405.

Problems

Solve the following problems with the help of logarithms. Arrange your work in systematic form for logarithmic computation, and, when possible, write the characteristics before referring to the table.

1. The human heart beats 75 times a minute. How many times does it beat in an ordinary life of 65 years? (Use 365 days to the year.)

2. Find the area of a circle of radius 14.36″. $A = \pi r^2$. Express π to four-figure accuracy. Why?

3. The volume of a right circular cone is given by the formula $V = \frac{1}{3}\pi r^2 h$. Find V when $r = 24.2''$ and $h = 14.9''$. What value of π must you use? Why?

4. How many gallons will a cylindrical tank hold if its radius is 6.12 ft. and its height 13.0 ft.? There are 7.48 gallons to a cubic foot.

5. A sphere is to be constructed to contain 183 cu. in. What inside radius should be used? (Solve for r the formula $V = \frac{4}{3}\pi r^3$ and proceed as indicated by your new formula.)

***6.** Using the formula $A = p(1 + r)^n$, find the amount, A, of \$2500, p, at 4%, r, interest compounded annually for 10 years, n.

***7.** In the preceding problem, find the amount if the interest is compounded semi-annually. (Notice that in this case $r = .02$ and $n = 20$. Work to the limit of the accuracy of your tables.)

***8.** Find to four-figure accuracy the amount of \$6200 at 4% interest for one year, compounded annually; semi-annually. Which answer is the larger? Why?

9. If $V^2 = 2\,gs$, $V = 418.0$, and $g = 32.16$, find s.

10. If $l = ar^{n-1}$, $a = 32.1$, $r = 5$, and $n = 8$, find l to the nearest third figure.

Using the formula of the preceding example, and considering a as approximate and r and n exact, find l if:

11. $a = .534$, $r = 3.5$, $n = 9$

12. $a = 13.49$, $r = 0.45$, $n = 12$

13. In a right triangle $a = \sqrt{c^2 - b^2}$. Find a if $b = 32.7'$ and $c = 58.9'$. (First express the quantity under the radical as the product of two factors and tell why this gives a more convenient form for logarithmic computation.)

14. In a right triangle the hypotenuse is 138′ and another side is 105′. Find the third side.

15. The area of a triangle is given by the formula $T = \sqrt{s(s-a)(s-b)(s-c)}$ in which s is one half the perimeter,

or, $\frac{1}{2}(a+b+c)$. Find T for the triangle in which $a = 48.7'$, $b = 53.8'$, and $c = 61.2'$. (First list the values of s, $s - a$, etc.)

***16.** The area of an oblique triangle is given by the formula $T = \frac{1}{2}ab \sin C$. If $a = 38.91''$, $b = 41.36''$, and $C = 23.87°$, find T.

***17.** Find the speed at the rim of a wheel 26.0″ in diameter making 315 *RPM* (revolutions per minute). Give answer in feet per second.

***18.** How long will it take \$5000 to amount to \$8000 at 5% compounded annually? (In solving you may find it helpful to take the logarithm of each side of one of your equations.)

***19.** How long will it take \$100 to double itself at $4\frac{1}{2}$% compounded annually?

***20.** The area of a circular ring between two concentric circles is given by the formula $A = \pi r_1^2 - \pi r_2^2$, where r_1 is the radius of the large circle and r_2 the radius of the small circle. Find the area of a ring when $r_1 = 48.9''$ and $r_2 = 21.7''$. (Factor before substituting.)

***21.** The area of an equilateral triangle is $\frac{b^2}{4}\sqrt{3}$ where b is the side of the triangle. Find the area of an equilateral triangle when $b = 13.42$ in.

Test. The Understanding and Use of Logarithms

1. Of what use are logarithms?

2. If log 32 = 1.5051 (to the base 10), express 32 as a power of 10. If $10^{3.9201} = 8320$, what is the logarithm of 8320 to the base 10?

3. What is the characteristic of the logarithm of:
729? 46.73? 0.9843? 0.0024? 467,000? 4.27×10^8?

4. Write an example which can be solved by the formula

$$\log \text{result} = \log 532 - \tfrac{1}{2}\log 127.$$

***5.** Is $\dfrac{\log 256}{\log 125} \equiv \log \dfrac{256}{125}$? Justify your answer.

6. What is the sign of the product of -3.16×8.71? What is the sign of the characteristic of the logarithm you would obtain in finding this product? What is the sign of the mantissa?

7. Supply the missing numbers: $\log 0.3824 = (\quad)$; $\log(\quad) = \bar{5}.2367$ (or $5.2367 - 10$).

8. Write a formula for the logarithmic evaluation of the following expression: $\dfrac{\sqrt{ac\sqrt{b}}}{e^2}$. Arrange the work for logarithmic computation; that is, do all the work you would do before turning to the tables.

Solve with the help of logarithms:

9. $\dfrac{\sqrt{167.5}}{0.09853}$ **10.** $\left(\dfrac{0.09436}{30.849}\right)^{\frac{1}{2}}$ **11.** $\sqrt{\dfrac{1.73 \times (41.0)^2 \times 300}{15.5}}$

12. Find b if $b = \sqrt{(c-a)(c+a)}$ and $c = 24.73$, and $a = 13.64$.

13. If $\log 35 = 1.5441$, what is $\log \frac{1}{35}$?

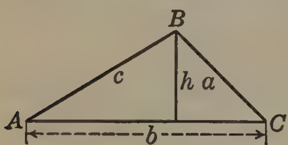
14. Mention methods of avoiding errors in long computations.

Algebraic Proofs

Before studying the proofs of two laws of logarithms to be found on page 130, it may be well to think about the subject of proof in algebra. Proving a formula consists in showing that the formula follows from previously known relations. Such reasoning from the known to the unknown plays an important part not only in algebra but in thinking in general.

Whenever we transform a formula, we reason systematically about the numbers involved in it; we prove that if the original formula is true, then the resulting formula is also true. Evidently the algebraic symbols are helpful not only in setting down the argument, but also in making the reasoning systematic and effective.

Explain and complete the following proofs:



*1. Given the formula $T = \frac{1}{2}bh$, for the area of a triangle, prove that $T = \frac{1}{2}ab \sin C$.

Plan of proof: $h = a \sin ?$ etc.

*2. Prove that in an oblique triangle with acute angles $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$.

Plan of proof: In the figure above, $h = a \sin ?$ or $h = c \sin ?$ etc.

*3. Prove that any number is divisible by 5 if its units digit is 0 or 5.

Plan of proof: Represent the number by $1000a + 100b + 10c + d$. Obviously the first three terms are divisible by 5, and the number will be divisible by 5 if $d = 0$ or $d = 5$.

*4. Prove that any number is divisible by 3 if the sum of its digits is divisible by 3.

Plan of proof: Represent the number as in the preceding example. Rewrite it as $999a + a + 99b + b + 9c + c + d$. Which terms are divisible by 3?

*5. Prove a theorem for division by 9 similar to that above.

*6. If $s = \frac{n}{2}(a + l)$ and $l = a + (n - 1)d$, prove that

$$s = \frac{n}{2}[2a + (n - 1)d].$$

*7. If $s(r-1) = ar^n - a$, prove that $s = \dfrac{a - ar^n}{1 - r}$.

*8. If $\dfrac{a}{b} = \dfrac{c}{d}$ prove that $\dfrac{a \pm b}{b} = \dfrac{c \pm d}{d}$. (What number must be added to and subtracted from both sides of the given equation?) Give a numerical illustration.

Concerning the Theory of Logarithms

*1. **Exponential and logarithmic symbolism.** The equation

① $10^3 = 1000$ may be written

② $3 = \log 1000$ to the base ten (explain)

or ③ $3 = \log_{10} 1000$.

Equations ② and ③ are both read in the same way. Show how they may be obtained from ①. More generally, the equation $a^l = n$ may be written to read $l = \log n$ to the base a, or, more briefly, $l = \log_a n$. The first of these equivalent equations is said to be in exponential form; the third, in logarithmic form.

Write in logarithmic form:

*2. $10^4 = 10{,}000$ $\quad 10^{-1} = 0.1$ $\quad 10^0 = 1$ $\quad 10^{.5} = 3.16$

*3. $10^{2.8751} = 750$ $\quad 10^{0.8306} = 6.77$ $\quad 10^{\bar{1}.4440} = 0.278$

*4. $10^a = b$ $\quad 10^x = c$ $\quad B = s$ $\quad B^x = y$

*5. With the help of the table of logarithms continue the drill of example 3 as far as may be necessary.

Write in exponential form:

*6. $\log_{10} 100 = 2$ $\quad \log_{10} 100{,}000 = 5$ $\quad \log_{10} 1 = 0$

*7. $4 = \log_2 16$ $\quad 5 = \log_2 32$ $\quad 2.8791 = \log_{10} 757$

*8. $l = \log_a b$ $\quad s = \log_y x$ $\quad \log_b n = c$

Two theorems concerning logarithms.†

***9.** *Theorem:* The logarithm of a product is the sum of the logarithms of the factors.

Given any two positive numbers, M and N, to prove: $\log_b MN = \log_b M + \log_b N$.

Proof: Let ① $\log_b M = l_1$

② $\log_b N = l_2$

Then ③ $M = b^{l_1}$ — ① and ② written in exponential form.

④ $N = b^{l_2}$

⑤ $MN = b^{l_1+l_2}$ — ③ × ④ by law of exponents.

⑥ $\log_b MN = l_1 + l_2$ — ⑤ written in logarithmic form.

⑦ $\log_b MN = \log_b M + \log_b N$ — ① and ② substituted in ⑥.

***10.** *Theorem:* The logarithm of any power of a number is equal to the logarithm of the number multiplied by the exponent of the power. Given any positive number, N, to prove: $\log_b N^p = p \log_b N$.

Proof: Let ① $\log_b N = l$

Then ② $N = b^l$ — ① written in exponential form.

③ $N^p = b^{pl}$ — ② raised to the pth power.

④ $\log_b N^p = pl$ — ③ written in logarithmic form.

⑤ $\log_b N^p = p \log_b N$ — ① substituted in ④.

Notice that to *prove* these theorems means to show that they depend upon the laws of exponents. The plan of proof is to translate the theorem into exponential form, to perform the required operation, and to translate back into logarithmic form.

***11. The logarithmic graph.** On page 262 may be found a graph which shows numbers on one scale and their logarithms on the other scale.

By reference to the table of logarithms you could construct

† Proof of these two theorems is listed in the requirements of the College Entrance Examination Board.

such a graph. As the graph suggests, the change in the value of the logarithm is continuous as the number changes. Are changes in the logarithm proportional to changes in the number?

***12. Finding the logarithms of numbers.** If you did not have tables of logarithms, you could find approximate values graphically. To do so you would find the logarithms of certain numbers, plot the corresponding points of the graph, and draw a smooth curve through these points. For example,

$10^{.5} = 10^{\frac{1}{2}} = \sqrt{10} = 3.16;$ therefore $\log 3.16 = .5$

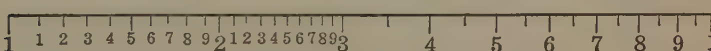
$10^{.25} = 10^{\frac{1}{4}} = \sqrt[4]{10} = \sqrt{\sqrt{10}} = 1.78;$ therefore $\log 1.78 = .25$

$10^{.75} = 10^{\frac{3}{4}} = (\sqrt[4]{10})^3 = ?^3 = 5.62$, etc.

$10^2 = 100;$ therefore $\log 100 = 2$, and so on.

Suggest other numbers of which you can find the logarithms. Show how the approximate values of the logarithms of other numbers could be read from the graph.

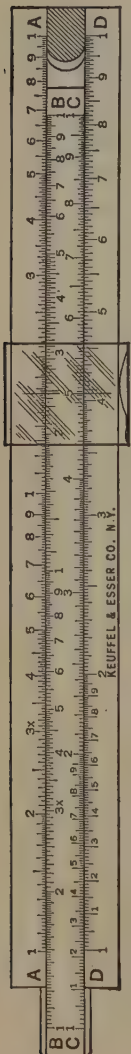
***13. The logarithmic scale and the slide rule.** The scale pictured below is a logarithmic scale; that is, the spaces are pro-



portional not to the numbers written on the scale, but to their logarithms.

Study the scale. Is 9 twice as far from the left end of the scale as 3 is? Is the log of 9 twice the log of 3? Explain.

A slide rule (see illustration on page 132) has two equal logarithmic scales, *A* and *B*, and these are arranged so as to slide one along the other. Point out these two scales. Show how to add two logarithms with the help of the sliding scales. Adding two logarithms is equivalent to performing what operation upon the numbers? You can learn how to use a slide rule without other help than that given in a booklet of instructions.



*14. What do you understand by Kepler's statement that "logarithms may be said to treble the effective life of an astronomer"? Look up Napier.

PART III. LOGARITHMS APPLIED TO TRIGONOMETRY

Introduction

Questions for discussion

1. In what processes in the solution of problems in trigonometry do you think that logarithms will be helpful?

2. If you are to use logarithms in trigonometric computation, you will need tables of logarithms of what numbers?

3. What is the sign of the characteristic of a number which is less than 1? What is the sign of the characteristic of the logarithm of the sine of any acute angle? Why? Can you make a similar statement about the cosine? about the tangent? Turn to the tables of logarithms of trigonometric functions on pages 484–87 or those on pages 492–95. Notice that the characteristics of these logarithms are printed in the tables except that -10 is omitted in such a logarithm as $9.4130 - 10$. Verify the answers you have just given.

4. Illustrative example. In a right triangle $A = 19.70°$, $c = 426.7'$. Find a.

Plan of solution:

① $a = c \sin A$ Explain how this equation is derived from the definition of $\sin A$.

② $\log a = \log c + \log \sin A$ This is the formula for the logarithmic computation. How does the procedure differ from that in the solution without logarithms?

③ $\log c$, or $426.7 = 2.6301$

④ $\log \sin A$, or $19.70° = \bar{1}.5278$ Verify this value in two ways: (*a*) find sin 19.70° in the table and look up the logarithm of this number: (*b*) find log sin 19.70° in the table of logarithms of sines.

⑤ $\log a$ or ( ) = ______

Complete. Check by computing the value of $\sin A$ from the values of a and c.

The Use of Logarithmic Tables of Trigonometric Functions

1. Verify the following:

$\log \sin 15.3° = \bar{1}.4214$; $\log \cos 15.3° = \bar{1}.9843$;
$\log \tan 15.3° = \bar{1}.4371$; $\log \sin 22.34° = \bar{1}.5799$;
$\log \cos 48.37° = \bar{1}.8224$; $\log \tan 69.06° = 0.4172$;

$$\log \sin 10° 58' = 9.2793 - 10.$$

Find logarithms of sines, cosines, and tangents of the following angles. Study particularly the interpolation in the tables of logarithms of cosines.

2. 38.53° **3.** 27.18° **4.** 49.15° **5.** 73.05° **6.** 82.89°
7. 19° 17′ **8.** 45° 29′ **9.** 59° 59′ **10.** 82° 44′

Supply the missing numbers:

11. $\log \sin (\quad) = \bar{1}.9842$ Ans. 74.65° or 74° 38′

12. $\log \cos (\quad) = \bar{2}.9300$ **13.** $\log \cos (\quad) = \bar{1}.7982$

14. $\log \tan (\quad) = 0.0081$ **15.** $\log \tan (\quad) = 1.0783$

16. Comment on interpolation in the first and last lines in the tables. Why is it not always reliable?

Try Exercise 39, page 407.

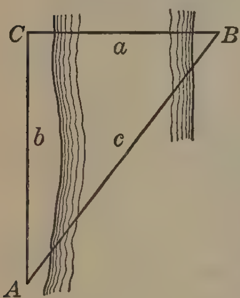
Problems

It is a good idea roughly to check your solutions by means of scale drawings. Squared paper is helpful in making such constructions. For each example use some rough check and some more accurate check. Solve with the help of logarithms:

1. Solve the right triangle in which $a = 43.21'$ and $A = 54.35°$ ($54° 21'$). See example 4, page 132.

2. The shadow of a tree is 24.0 ft. long when the angle of elevation of the sun is 48.2°. What is the height of the tree?

3. In preparing to build a bridge, the distance from C to B (see sketch) must be measured. For this purpose what line must be measured with a tape and what angle must be measured with a transit? If $b = 273$ ft. and $A = 37.3°$, find a. Find c, and check your work by means of the formula $a = \sqrt{c^2 - b^2}$.



4. Solve the preceding problem if $b = 489.6'$ and $A = 26.09°$ ($26° 5'$).

5. If $b = 49.73'$ and $A = 18° 40'$.

6. From a point A on the rim of the Grand Cañon of the Colorado a point B at water level in the bottom of the cañon is observed at an angle of depression of 18.30°. The horizontal distance between C and B is 15,000 ft. What is the depth of the cañon? How far is it from A to B? Check as in problem 2.

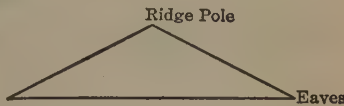
7. Solve the preceding problem if the angle of depression is $19° 24'$ and the horizontal distance is 14,840 ft.

8. Five equally spaced holes are to be bored on a circular steel plate. Their centers are to lie on a circle of radius 8.37″. Find for the machinist the distance between centers.

9. Solve the preceding problem if there are seven holes and the radius of the circle through their centers is 11.24″.

10. A road rises at an angle of 3.33° with the horizontal; how far does the road rise in 100 ft. measured along the road?

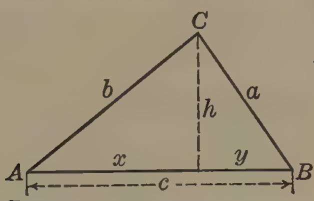
11. The height of the ridge pole above the eaves is 12 ft. and the distance between the eaves is 48 ft. The roof rises at what angle with the horizontal?



*12. Find the area of an oblique triangle in which $a = 418'$, $b = 298'$ and $C = 43.1°$ (43° 6′). $T = \frac{1}{2} ab \sin C$.

13. Find the area of a triangular building lot of which one side is 78.14′, another 87.32′, and the included angle 73°14′.

14. Find the missing parts and the area of the triangle shown



in the accompanying sketch. Check your solution. $A = 38.7°$, $c = 68.5'$, $b = 57.9'$, h is perpendicular to c.

*15. In an oblique triangle, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$. Solve the formula for a and find a when $A = 42.3°$, $B = 67.4°$, and $b = 14.3'$. Find the area of the triangle.

Using the formula of the preceding problem, supply the angles and distances and areas missing below.

*16. $a = 37.19'$, $b = 26.18'$, $A = 65.34°$ (65° 20′).

*17. $a = 46.91'$, $b = 22.56'$, $B = 18.91°$ ($18° 55'$).

Try Exercise 40, page 407.

(If time permits and more practice is needed, solve by the help of logarithms problems 2–11 on pages 88 and 89.

Tests

Re-read paragraph 5 of *How to Study Algebra*, page xx.

Test A

1. In order that the laws of positive exponents may hold for fractional, negative, and zero exponents, what meaning must be given to x^0, $y^{\frac{1}{2}}$, and z^{-5}? Show that your second answer is correct.

2. Give results in simplest form:

$$a^{n-1} \times a \qquad \frac{a^{n-1}}{a} \qquad 8^0 + 8^{\frac{1}{2}} \cdot \sqrt{2} + 5^{-1}$$

Write without negative exponents:

$$\frac{2\,a^2b^{-2}}{ac^{-3}} \qquad \frac{x^2 - y^{-2}}{x - y^{-3}}$$

3. (*a*) Write by powers of 10: 93,200,000; 0.000,008,342

(*b*) What is the characteristic of the logarithm of each of the numbers in (*a*)?

(*c*) If $\log\ 2.384 = 0.3773$, what is the logarithm of 2.384×10^3?

4. If $\log\ 9 = 0.9542$ and $\log 18 = 1.2553$, it follows that $\log 2 = ?$ State a general law that justifies the reasoning used above. $3 = 10^{0.4771}$ and $7 = 10^{0.8451}$. Write each of the preceding statements in logarithmic form and show how to find the logarithm of 21 without referring to the table.

5. Prove that the law, $\log_b MN = \log_b M + \log_b N$, follows from a law of exponents. Write the proof formally, step by step.

***6.** State in symbols and prove (as in the preceding example) the law for raising a number to a power by logarithms.

***7.** Find the amount of \$3500 compounded annually for 8 years at 4%. Use the formula $A = p(1 + r)^n$. Give the result to as many significant figures as your tables permit.

8. If the logarithm of 2.147 is 0.3318, what is the logarithm of 2.147×10^2? of 214.7? of 214700?

***9.** Do the numbers $\bar{1}.3267$ and -1.3267 have the same values? Explain your answer. Supply the missing numbers $\bar{3}.4263 = -2.????$

Find the value of:

10. $\dfrac{13.52^2}{4.573}$

11. $\dfrac{3.426^{\frac{1}{2}} \times 5.297^2}{32.56^3 \times 0.8631}$

12. $\dfrac{98.73\sqrt{42.86}}{\cos 42°17'}$

13. $\dfrac{0.9244 \times \sqrt[5]{296.666}}{0.01764}$

14. Estimate the value of each of the expressions in 10, 11, 12, and 13 and show how you reached each estimate.

15. In a right triangle, $a = 24.16'$, and $B = 48°\ 22'$. Find c.

16. Divide $x - 4\,x^{\frac{1}{2}}b^{-\frac{1}{3}} - 5\,b^{-\frac{2}{3}}$ by $x^{\frac{1}{2}} - 5\,b^{-\frac{1}{3}}$. Check the result by multiplication; also check by setting $x = 4$ and $b = 8$ in dividend, divisor, and quotient.

Test B. Equations

Solve and check:

1. $\dfrac{x + 9}{6} - \dfrac{2\,x - 3}{12} + \dfrac{x - 4}{x + 7} = 0$

2. $4\,x^2 + x = 3$

***3.** $0.728\,x^2 - 1.03\,x - 41.4 = 0$ (The numbers in this example are approximate.)

4. If (1) $a^3 - ad^2 = 120$, and (2) $a - d = 5(a + d)$, solve (2) for d, substitute this result in (1), and solve the resulting equation for a.

5. If (1) $x + 3y = 25$, and (2) $x^2 + 3y = 115$, solve (1) for x, substitute the result in (2), and solve the resulting equation for y.

6. $\dfrac{c}{m + x} = \dfrac{m}{c - y}, \qquad \dfrac{c}{m + y} = \dfrac{m}{c - x}$

Test C. Problem Solution

In solving use a tabular form of arrangement. See, for example, pages 45 and 49.

1. A man sets out to walk 24 miles. He walks for three hours and then decreases his average speed by one mile an hour. The trip requires seven hours. How long would it have taken him if he had maintained the average speed at which he set out?

2. A photographer has two mixtures of a certain chemical and water, the one containing 50% chemical and the other only 10%. He needs a sixteen-ounce solution which shall contain 25% chemical. How much of each mixture shall he use?

3. The turnstiles at the entrance to an industrial exposition showed a total attendance of 16,104 persons; the ticket receipts were $7271. If the charge for adults was 50 cents and for children 25 cents, how many adults attended the exposition?

*4. A silver-plated lead cup weighs 18.40 oz. in air and only 16.78 oz. in water. What is the actual amount of silver in the cup if one ounce of silver weighs only 0.905 oz. in water and one ounce of lead weighs 0.913 oz. in water?

5. A man wishes to divide $5000 between two bond investments that pay $4\frac{1}{2}$% and $6\frac{3}{4}$% respectively in such a way that he will receive equal incomes from the two investments. How shall he divide the money?

Test D. Meaning and Use of Symbols

1. Does $4^{\frac{1}{2}} = (2^2)^{\frac{1}{2}}$? **2.** Does $\sqrt{9} = 3$ or -3 or ± 3?

3. Replace the radical signs by fractional exponents:

$$\sqrt{5} \quad \sqrt{32} \quad \sqrt[3]{16} \quad \sqrt[4]{9} \quad \sqrt{2^3} \quad 2\sqrt{2} \quad 7\sqrt[5]{7} \quad a\sqrt[3]{a^{\frac{3}{2}}}$$

4. Replace the fractional exponents by radical signs:

$$5^{\frac{3}{2}} \quad 2^{\frac{2}{3}} \quad 3^{\frac{1}{3}}4^{\frac{2}{3}} \quad \left(\tfrac{4}{3}\right)^{\frac{1}{2}} \quad 3^{\frac{1}{6}}7^{\frac{1}{3}} \quad \left(\tfrac{1}{4}\right)^{\frac{1}{2}}(3)^{\frac{1}{2}} \quad (a^5)^{\frac{1}{4}}$$

5. To "simplify" a radical expression as defined in Chapter I, means to leave (*a*) ——, (*b*) ——, (*c*) ——. (Page 65.)

6. Rationalize the denominators:

$$\frac{3\sqrt{2}}{\sqrt{5}} \qquad \frac{3+\sqrt{2}}{\sqrt{5}} \qquad \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{5}}$$

Rewrite the preceding expressions using fractional exponents instead of radicals and then transform them so as to check your answers.

***7.** Rewrite with fractional exponents and multiply: $\sqrt{3} \times \sqrt[3]{2}$. Express the result in radical form.

***8.** Which symbolism do you consider more efficient and why, radicals or fractional exponents?

9. Find the value of $16^{\frac{3}{4}}$. Which order of performing the operations is the shorter?

10. What is the value of $4^{\frac{1}{2}}$? Is $(9+3)^{-1} \equiv \frac{1}{9} + \frac{1}{3}$?

11. Does $(a - 2b)^2 = (2b - a)^2$? Explain.

12. Does $3a^0 = 1$ or 3? Explain.

13. Does $\dfrac{2x^{-2}y}{z} = \dfrac{2y}{x^2z}$ or $\dfrac{y}{2x^2z}$?

14. What is the meaning of x^b?

15. What is the meaning of $x^{\frac{3}{2}}$? Why is it given this meaning?

16. Give at least one use for literal exponents.

17. State in symbols four laws of exponents.

18. State in symbols four laws for logarithmic computation.

19. Tell why we "extend the meaning of exponents" and how.

***20.** Tell how a theoretical study made in this chapter led to practical results.

21. If $\log 2 = 0.3010$ and $\log 6 = 0.7782$, find without reference to the table, $\log 12$; $\log 3$; $\log 36$; $\log \frac{1}{3}$; $\log \frac{1}{9}$; $\log \frac{1}{6}$.

22. Find by logarithms the reciprocal of 0.03761. Check your answer.

23. What is the value of $\log_2 16 + \log_{10} 100 - \log_3 9$?

***24.** What are the limiting values of numbers having zero for the characteristic of their logarithms? As a number grows from 0.5 to 5, what change takes place in its logarithm?

Test E. Products and Factors

Give the products of:

1. $(5\,xy^2 + 1)(5\,xy^2 - 1)$

2. $(5\,xy^2 - 1)^2$

3. $(x^3 + 3\,x)(x^3 - 3\,x)$

4. $(3\,x^a + 5\,x^b)(2\,x^a - 3\,x^b)$

5. $(x^{a-b} - y^{a+b})^2$

6. $(-3\,x + 2\,a)(-3\,x - 2\,a)$

Factor:

7. $2\,x^2 - 17\,x + 8$

8. $3\,x^2 + 2\,xy - 16\,y^2$

9. $3\,x + 6 - 2\,x(x + 2)$

10. $4\,x^2 - 25$

11. $x^2 - (a - c)x - ac$

12. $x^{2m} + 2\,x^m y^n + y^{2n}$

13. $m(m + 1)(m + 3) - (2\,m + 6)$

CHAPTER IV

THE PROGRESSIONS

Introductory study. Examine each sequence below and supply the next two terms at the right, and if possible the next two terms at the left.

1. $3, \quad 6, \quad 9$	**2.** $19, \quad 11, \quad 3$	**3.** $1, \quad -3, \quad 9$
4. $1.7, \quad 2.0, \quad 2.3$	**5.** $\frac{5}{12}, \quad 1\frac{2}{3}, \quad 6\frac{2}{3}$	**6.** $25, \quad 36, \quad 49$
7. $\sqrt{3}, \quad 3, \quad 9$	**8.** $-\frac{3}{2}, \quad 1, \quad -\frac{2}{3}$	**9.** $\frac{1}{4}, \quad \frac{1}{5}, \quad \frac{1}{6}$
10. $2, \quad 2, \quad 2$	**11.** $\frac{1}{16}, \quad \frac{1}{25}, \quad \frac{1}{36}$	**12.** $24, \quad -2, \quad \frac{1}{6}$
13. $4.5, \quad 5.6, \quad 6.7$	**14.** $1, \quad \frac{1}{2}, \quad \frac{1}{4}$	

15. $\frac{a}{b}, \quad a, \quad ab$ **16.** $a^3, \quad a^2, \quad a$ Use negative exponents.

17. $m^{\frac{3}{4}}l^{\frac{1}{4}}, \quad m^{\frac{1}{2}}l^{\frac{1}{2}}, \quad m^{\frac{1}{4}}l^{\frac{3}{4}}$ **18.** $\sqrt{7}, \quad \sqrt{9}, \quad \sqrt{11}$

19. $x^2, \quad x^{\frac{4}{3}}, \quad x^{\frac{2}{3}}$ **20.** $180, \quad 360, \quad 540$

21. $\sin 1°, \quad \sin 3°, \quad \sin 5°$ **22.** $a_1, \quad a_2, \quad a_3, \ldots, a_n$

Preliminary definitions. A *sequence* is a succession of numbers which proceed according to some fixed law. A *series* is the indicated sum of the numbers forming a sequence. Either a sequence or a series may be called a *progression*. In this chapter two kinds of progressions are studied. One is called *geometric* and the other *arithmetic*.

Aims of the chapter. Two results of studying this chapter should be, (1) an understanding of the laws of geometric and arithmetic progressions and their applications, and (2) more important, an understanding of the scientific method of work. The steps in a scientific mathematical study are these: (1) Find the numbers involved and their relations; (2) state these rela-

tions by means of formulas; (3) apply these formulas to situations in which they are useful.

Part I. Arithmetic Progression

The situation. When a boy went to work for a certain farmer, he received board and lodging, and, for the first month \$20, for the second \$22, for the third \$24, and so on. What was his monthly increase in wages? How much cash should he receive for the fifth month? for the twelfth? for the first year? If you knew only what he received as a constant monthly increase and that his cash wages were \$20 for the first month and \$24 for the third, how could you find his cash wages for the second?

The analysis. The numbers involved are 20, 22, 24, and so on. They are so related that each one of them beginning with the second may be obtained from the one before it by the addition of a certain number, called the *constant* (or common) *difference.* Such a sequence is called an *arithmetic sequence.* Define it. On page 141, examples 1–20, point out arithmetic sequences and give the constant difference in each. Write an arithmetic sequence; an arithmetic series. Notice that either one of them may be called an arithmetic progression.

The boy, wishing to find his total cash wages for the year, wrote $20 + 22 + 24 + \dots$ This indicated sum is an arithmetic series. We want to find a formula which will give the sum of such a series by a shorter method than that of adding all the terms, and another formula which will permit us to write down any term without finding all the preceding terms. You can probably develop both of these formulas for yourself.

The numbers and their symbols.

		Symbol
1. The first term,	20	a
2. The common difference,	2	d
3. The number of terms,	12	n
4. The last, or nth, term,	42	l
5. The sum of the terms,	?	s

General conclusions and formulas.

Any arithmetic series may be written in the form:

$$\underset{\text{1st}}{a} + \underset{\text{2d}}{(a+d)} + \underset{\text{3d}}{(a+2\,d)} + \underset{\text{4th}}{(a+3\,d)} + \dots + \underset{n\text{th}}{a+(n-1)d}$$

Notice that in each term the coefficient of d is one less than the number of the term. How is the formula for the nth term derived? State this formula in words. Notice how the literal symbols of algebra are helpful in making such general statements.

$$\textbf{Formula I.}\quad l = a + (n-1)d$$

The formula for the sum of the terms of an arithmetic series may, of course, be written:

$$s = a + (a+d) + (a+2\,d) + \dots + (l-2\,d) + (l-d) + l$$

This formula, however, is not a convenient one to use. Why? To replace such a formula by one more compact and easy to use is one of the constantly recurring tasks of mathematics. There is no general method for doing it. In this case, the key to the method lies in observing that the sum of the first and last terms is the same as the sum of the second and next to the last, and so on. Verify this. Now rewrite the foregoing formula thus:

① $s = a + (a+d) + (a+2\,d) + \dots + (l-2\,d) + (l-d) + l$

② $s = l + (l-d) + (l-2\,d) + \dots + (a+2\,d) + (a+d) + a$

③ $2\,s = (a+l) + (a+l) + (a+l) + \dots$ ① + ②

Hence, $2\,s = n\,(a+l)$

$$\textbf{Formula II.}\quad s = \frac{n}{2}\,(a+l)$$

Substituting Formula I in Formula II gives

$$\textbf{Formula III.}\quad s = \frac{n}{2}[2\,a + (n-1)d]$$

To sum up.

(*a*) What is the difference between a sequence and a series? (*b*) What is an arithmetic progression or A.P.? (*c*) State a rule for finding the constant difference in an arithmetic progression. (*d*) What three steps in a scientific mathematical study are mentioned on page 141? (*e*) Give three formulas for arithmetic progressions and tell what purpose each serves. Derive each formula. (*f*) How does algebra usually state general conclusions?

The Use of the Formulas

1. What are the steps to take in using the formula method of problem solution?

2. Find the sum of the first 20 odd numbers, (*a*) by addition, and (*b*) by means of the formula for s. Which method is easier? Which gives more opportunity for error? Find, without pencil, the last term; the average of the first and last terms; the sum of the series.

3. Find the 8th term of the sequence 7, 4, 1 . . . Observe that $n = 8$.

4. Which term of the A.P. (Arithmetic Progression) 1, 5, 9, . . . is 37? (Formula I.)

Try Exercise 41 A, B, C, D, page 408.

5. Indirect use of the formulas. An article was bought on the installment plan, and interest on unpaid balances was paid in the following amounts: \$1.92, \$1.84, \$1.76, . . . \$0.08. What was the total amount of interest paid?

Plan of solution: $a = 1.92$, $d = -.08$, $l = .08$, $s = ?$ Will any one of our formulas give a value for s in terms of a, d, and l? If not, we must use two formulas, either in succession or as a pair of equations in two unknowns. Find n by the use of Formula I, then substitute in which Formula?

6. A debt is to be paid off in installments as follows: \$200, \$195, \$190, ... \$5. Find the total amount to be paid.

7. A man puts \$0.25 in the bank one week and \$0.50 the next week, and so on until he reaches \$10 a week. How much money has he deposited, and over a period of how many weeks do the deposits run?

8. What term of the sequence 6, $8\frac{1}{2}$, 11, ... is 31?

9. Write the seventh and the eleventh terms of the sequence $a, a + d, a + 2d, \ldots$ If the seventh term is 12 and the eleventh term is 18, find a and d.

10. Find d in an A.P. of which the third term is 3 and the ninth is 9.

11. Find d in an A.P. of which the twenty-first term is -10 and the ninth is 50.

12. Find d in an A.P. of which the fourth term is $5x + 6y$ and the seventh term is $8x + 12y$.

13. Insert the three missing terms in the sequence 5, ?, ?, ?, 17.

Plan of solution: $a = 5$, $n = 5$, $l = 17$. If we knew d, we could supply the missing terms. Which formula is indicated?

14. Insert the four missing terms in the sequence 38, ?, ?, ?, ?, -17.

Arithmetic means.

15. In an arithmetic sequence of three terms the middle term is the *arithmetic mean* between the other two terms. Write such a sequence in the general form $a, a + d, a + 2d$, and show that the arithmetic mean between the two end terms is the *average* of the two. In the arithmetic sequence 8, ?, 14, find the missing term by formula as in example 13 and show that it is the average of 8 and 14. The expression *arithmetic mean* is more general

than *average* because it can be extended to include several terms while *average* includes only one. How many arithmetic means were inserted in the sequence of example 13? of example 14? *In a series of n terms the $(n-2)$ terms between the first and the last are the arithmetic means between them.* Explain. If a progression has 10 arithmetic means between its first and last terms, how many terms has it in all?

16. Insert 5 arithmetic means between 3 and 33. (See example 15.) Insert 6 arithmetic means between -18 and 17.

Try Exercise 41, E, F, pages 411, 412.

Problems

It is possible to make a variety of problems by stating relations between the numbers involved in an arithmetic progression. Most of such problems are not practical, but they serve to test your ability to understand the relations stated and to use the formulas. More practical problems are those, for instance, of simple interest.

1. The sum of the means between the first and last terms of an A.P. is 132. The first term is 1 and the last is 21. How many means are there? What are the first two means?

Plan of solution: $a = 1$, $l = 21$, $s = 132 + 22 = 154$. Explain. $n = ?$, $d = ?$

① $s = \frac{n}{2}(a + l)$. Why is this formula selected? Next find d.

2. Find the fourteenth term of an arithmetic progression in which the fifth term is 11 and the ninth term is 7.

Plan of solution: Represent the fifth term by $a + 4d$ and the ninth by $a + 8d$; form a pair of equations and solve for a.

3. In the series $a - 2d$, $a - d$, a, $a + d$, $a + 2d$, show that the sum or the product of the first and last terms or of the

second and fourth terms give simpler expressions than the sum or the product of the first and second terms or of the fourth and fifth. How would you take advantage of this fact in solving the following problem? (In solving a pair of equations in two unknowns it is sometimes convenient to eliminate one letter by substituting in one equation its value found in the other equation.) The sum of the first and third terms of an arithmetic progression is 31, and their product is 228. Find the first three terms.

4. If you invest \$100 at the beginning of each year at 6% simple interest, how much will you have at the end of ten years?

Plan of solution:

I. *Find the A.P.* The first investment of \$100 would amount in 10 years to \$160; the second in 9 years to \$154; the third in 8 years to \$148. The progression is 160, 154, 148 . . .

II. Select the appropriate formula. Is s required or n?

$$s = \frac{n}{2}\left[2\,a + (n-1)d\right]$$

Complete and check. The law of arithmetic progression is sometimes called the simple interest law. Why?

5. A \$3000 mortgage was paid off by annual payments of \$150 with 6% interest. What was the total amount of the payments?

Plan of solution:

I. Write the progression.

II. To find n, divide 3000 by 150. Explain.

III. Decide whether s is wanted or l and choose the appropriate formula.

6. A man invests \$500 at the beginning of each year at 5% simple interest. How much has he to his credit at the end of 10 years? at the end of 20 years?

***7.** A man pays off a debt of \$24,000 by monthly payments of \$100 and interest. The first payment is made at the end of the first month and interest is 6% per annum. What is the total amount of interest paid while the debt is being cleared off?

8. A limited express increases its speed from 20 miles an hour to 75 miles an hour in eleven minutes. What is the rate of increase in miles per hour per minute?

Notice that if $a = 20$, then $n = 12$ and not 11, because we are counting from the *beginning* of the first minute to the end of the eleventh minute. Use the accompanying illustration to fix this

idea clearly in mind. Refer to it whenever necessary.

Try Exercise 41, G, page 412.

Tests. Arithmetic Progression

Test A. To Test Your Understanding

1. Define sequence; arithmetic series; common difference.

2. Tell how you recognize an arithmetic progression.

3. What are arithmetic means? On what condition is an arithmetic mean the same as an average?

***4.** Why is the law of increase in an arithmetic progression sometimes referred to as the law of simple interest?

5. Write four terms of an arithmetic progression of which the first term is a and the common difference d.

6. What are the three fundamental formulas of arithmetic progression? Derive them.

***7.** At the beginning of this chapter what steps of a mathematical study are illustrated? Which do you consider more important to remember, the results of such a study or the method of making it?

*8. Comment on the following statement from a book on biology: "Whenever any quantity grows in such a way that its increase or decrease in value during equal intervals of time is always a constant part of its original value, then the successive values attained by it at the ends of these intervals of time form the terms of an arithmetic progression."

Test B

1. Find the 23d term of $\frac{4}{9}$, 1, $\frac{14}{9}$ $\cdots$

2. Find the sum of $1.04 + 1.06 + 1.08$ to 15 terms.

3. From page 141 copy the first A.P. in the second column and find its tenth term.

4. How many terms of the series 1, 3, 5 ... must be taken to give a sum of 225?

5. Find the arithmetic mean between $\dfrac{a+b}{a-b}$ and $\dfrac{a-b}{a+b}$

6. Insert 5 arithmetic means between 2 and 20.

7. The product of 3 numbers in A.P. is 120, and the first number is five times the last. Find the numbers.

*8. A debt of $9000 is to be paid off in monthly installments of $100 and interest at 6%. Find the total amount to be paid.

9. Given the formulas $l = a + (n-1)d$ and $s = \dfrac{n}{2}(a + l)$, eliminate a and solve the resulting formula for d.

*10. How many terms of the series 4, 8, 12, ... must be taken if the sum of the first half is to be to the sum of the second half as 3:8?

*11. Derive a formula for the sum of the first n odd numbers. Check by numerical illustration.

PART II. GEOMETRIC PROGRESSION

Introduction

Re-read the *Aims of the Chapter*, page 141, and tell how you expect to proceed to study this new kind of progression. Plan such a study and carry it out with understanding. To be able to do this is probably the most important lesson to be learned in a second course in algebra.

The situation. An ancient problem asks how many grains of wheat will be needed if one grain is placed upon the first square of a chess board, 2 grains upon the second square, 4 grains upon the third, and so on for the 64 squares. Do these numbers form an arithmetic sequence? How do you know?

The analysis. In the sequence above how is each number obtained from the number before it? What is the difference between the first number and the second? the second and the third? What is the ratio of the second to the first? the third to the second? *A geometric progression is one in which each term after the first is obtained by multiplying the preceding term by a constant ratio.*

State a rule for finding the constant ratio in a geometric series. Tell how you will recognize a geometric sequence. Copy from page 141 those sequences which are geometric and extend each two terms to the right and two to the left.

The quantities and their symbols. Complete this list of the quantities to be considered and their symbols.

		Symbol
1. The first term,	1	a
2. The common ratio,	2	r
3.	64	n
4.		l
5.		s

Observe that we started with a particular problem, that we have already discovered the fundamental law relating the numbers

involved, and that we are about to construct formulas which may be applied to similar problems.

General conclusions. Formulas. The general geometric series may be written $a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$. Explain. How is the exponent of each term related to the number of the term?

Formula I. $\boldsymbol{l = ar^{n-1}}$

① $s = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1}$.

② $rs = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n$. ① × r

Explain. Here, as in our study of A.P. we want to reduce this formula to a simpler and more useful form. Can we do this exactly as we did before?

③ $rs - s = ar^n - a$ ② − ①

④ $s(r - 1) = ar^n - a$, or

Formula II. $\boldsymbol{s = \dfrac{ar^n - a}{r - 1}}$

⑥ $rl = ar^n$ (Formula I multiplied by r)

⑦ **Formula III.** $\boldsymbol{s = \dfrac{rl - a}{r - 1}}$ ⑥ substituted in ⑤

Comment on the method used to make the formula more compact. Show that Formulas II and III may be written $s = \dfrac{a - ar^n}{1 - r}$ and $s = \dfrac{a - rl}{1 - r}$. Show that Formulas II and III have no meaning when $r = 1$. Construct a G.P. in which $r = 1$, and show that its sum is na.

Look over the formulas of arithmetic and geometric progression and tell which two are well adapted to logarithmic computation and why.

Applications of the Formulas

1. Apply Formula II to the solution of the problem of the second paragraph on page 150.

Plan: $a = 1, \quad r = 2, \quad n = 64, \quad s = ?$

$$s = \frac{ar^n - a}{r - 1} = \frac{1(2)^{64} - 1}{2 - 1}$$

This is clearly a task for logarithms. Why? Find s to four significant figures. Notice that the -1 in the numerator has on the answer no effect whatever. With eight-place tables you could find its value to eight figures. If you wanted to find more figures than the tables permit, you could perform the necessary multiplications, but it would be somewhat laborious. The table of powers of 2 on page 112 might be used advantageously. Very large and very small numbers may well be written by powers of ten. See page 106.

2. In the problem on page 150, find the number of grains on the 10th square. (Here $a = 1, r = 2, n = 10, l = ?$ Which formula is indicated?) Find also the number of grains on the 5th, 7th, 8th, and 13th squares.

3. Find the seventh term and the sum of seven terms of the geometric series $5 + 50 + 500 \ldots$

4. Find by use of the formula the fourth term and the sum of four terms of $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ and of $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$

5. Find with the help of logarithms the sum of twelve terms of $3 + 18 + 108$.

Plan of work: $s = \dfrac{3(6)^{12} - 3}{5} = \dfrac{3}{5}(6)^{12} - \dfrac{3}{5}$

Show that the second fraction is too small to affect the result computed by logarithms.

6. Find with the help of logarithms the sum of nine terms of $25 + 5 + 1 + \ldots$ *Plan:*

$$s = \frac{25(\frac{1}{5})^9 - 25}{(\frac{1}{5}) - 1} \equiv \frac{25(\frac{1}{5})^9}{-\frac{4}{5}} + \frac{25}{\frac{4}{5}} \equiv \frac{125(\frac{1}{5})^9}{-4} + \frac{125}{4} \equiv \frac{(\frac{1}{5})^6}{-4} + \frac{125}{4}$$

Show that the first fraction is too small to affect the result computed by logarithms. Make an example in finding the sum of a G.P. in which sum the "first fraction" is negligible; one in which the "second fraction" is negligible.

7. Find the twelfth term and the sum of twelve terms of $4 - 1 + \frac{1}{4} \ldots$ indicate the results merely; do not simplify them.

8. Find by use of the formulas the fifth term and the sum of five terms of $2.2 + .66 + .198 + \ldots$ Check your results by writing the series and adding its terms.

Try Exercise 42, A, B, C, D, E, pages 415, 416.

Indirect Use of the Formulas

1. Find the sum of the series $15 + 60 + 240 + \ldots + 61{,}440$. What formula can be used in finding n? In finding s?

2. Find the eighth term of a G.P. (Geometric Progression) in which the first term is 1 and the fifth term 256. Which formula can be used in finding r? Show that r has two values.

3. Write the G.P. of which the fourth term is 24 and the eighth is 384.

Plan: ① $ar^3 = 24$ ② $ar^7 = 384$ Divide ② by ①

4. How many consecutive terms of the series $48 + 24 + 12 + \ldots$ make a total of $95\frac{1}{4}$?

5. In the sequence 2, 6, 18, ... which term is 486?

***6.** The population of a town increased from 10,000 in 1920 to 20,736 in 1924. If the yearly increase followed the law of geometric progression, what was r? At this rate of increase, what population could the town expect in 1929?

*7. A city grew from 78,330 in 1920 to 102,600 in 1929. If the yearly increase followed the law of geometric progression, what was r?

8. The first and third terms of a geometric progression are 2 and 18 respectively. Find the second term. Show that the middle term is $\pm\sqrt{al}$. Refer to Exercise 15, page 145, and suggest a name for a number, or numbers, inserted between two other numbers so as to form a geometric sequence.

Geometric means.

9. From a consideration of the preceding example, we conclude that the terms between the first and last term of a geometric sequence may be called "geometric means" between these two end numbers. When the sequence contains but three numbers, the middle number, or the geometric mean, is plus or minus the square root of their product; it is also called a mean proportional between the end terms. Contrast the arithmetic mean and geometric mean between 2 and 18. Give to each another name. How many geometric means are there between the end terms of a G.P. of ten terms?

10. Insert 5 geometric means between -27 and $-\frac{1}{27}$.

Plan of solution: $a = -27, \quad n = 7$ (why?), $\quad l = -\frac{1}{27}$.

We must find r if we are to write the other terms of the series. What formula is indicated? Is $r = \frac{1}{3}$ or $-\frac{1}{3}$ or $\pm\frac{1}{3}$? Are there two sets of means or one set?

11. Insert four geometric means between 1 and 243. Answer the questions of the preceding example. Is there more than one series?

Try Exercise 42 F and G, pages 417, 418.

Problems

In studying the following verbal problems: (1) If you are not

certain what the progression is, try to write its first three terms. (2) Make sure which unknown is required. (3) Make sure that you have the correct value for n; observe the "end cases" carefully. From the beginning of the first to the end of the fourth is 5 and not 4, just as there are five posts for four lengths of fence.

1. A blacksmith agreed to shoe a horse for 1 cent for the first nail. 2 cents for the second, 4 cents for the third, and so on; or, for $10. If there are 8 nails in each shoe, what will the first plan cost? Contrast with Problem 6, page 410.

2. A bacterium entered a human body, and at the end of 36 hours divided into 2 bacteria. By the end of another 36 hours each of these bacteria divided into two, and so on. Find the number of bacteria in the tenth generation; in the generation which came at the end of three weeks.

3. A rubber ball thrown to a height of 50 feet rises on each rebound one half of its previous height. How far has it traveled when it touches the ground for the sixth time?

Plan: Make a sketch of the path of the ball, $a = 100$ ft., the length of the first "round trip."

4. Solve the preceding problem if the word "thrown" is replaced by the words "dropped from." Make and solve two other similar problems.

***5.** Find the amount of $600 at 6% compounded semiannually for 5 years.

Plan: First find the G.P.

Term 1 Beginning of first half yr.	Term 2 End of first half yr.	Term 3 End of second half yr.	Term 4 End of third half yr.
600	600(1.03)	$600(1.03)^2$	$600(1.03)^3$

Study this sequence carefully. Show that the fourth term comes at the *end* of the third half year, etc. Hence, if $a = 600$ then $n = 11$ and not 10. See suggestion 3 above. Second, select the

formula. $a = 600$, $r = 1.03$, $n = 11$, $l = ?$ Show clearly that l is required, and not s. Since the data are exact, work to the degree of accuracy permitted by the table of logarithms. Notice that if 600 (1.03) is taken for a, then $n = 10$ and the last term, $600\,(1.03)^{10}$ may be written at once.

***6.** Find the amount of $700 at 4% compounded quarterly for six years.

***7.** Mrs. Brown placed $500 in a savings bank where the interest at 5% was compounded semi-annually. What will be the amount to her credit ten years after the deposit was made? 20 years after?

8. An automobile costing $2400 depreciates in each year 40% of its value at the beginning of that year. What is its value at the end of 4 years? 8 years? Answer to the nearest dollar.

***9.** A man deposits $300 at the beginning of each year and receives 5% interest compounded annually. How much has he to his credit at the end of the sixth year?

Plan of solution: The last $300 deposited remained at interest one year and amounted to $300 (1.05). The next to the last deposit amounted to $\$300\,(1.05)^2$. The progression is $300\,(1.05) + 300\,(1.05)^2 + 300\,(1.05)^3 + \ldots$ $a = 300\,(1.05)$, $r = 1.05$, $n = 6$. s is wanted. Explain.

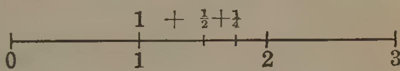
***10.** A man deposited $500 every six months in a savings bank that paid $4\frac{1}{2}$% interest compounded semi-annually. What was his account worth at the end of 5 years?

Try Exercise 42 H, page 418.

Infinite Series

1. Introduction. Can a progression have an infinite number of terms; that is, can it be unending? Consider for instance the A.P. $1 + 2 + 3 + 4 + \ldots$ or the G.P. $1 + 4 + 16 + 64 + \ldots$

Does the sum of n terms of either of these series increase without limit as the number of terms is increased? Can you make the sum of n terms greater than 1,000,000 by taking a sufficient number of terms? Can you make the sum greater than any previously assigned number by taking n sufficiently large? Consideration of the series $1 + \frac{1}{2} + \frac{1}{4} + \ldots$ leads to a different and interesting conclusion. Find its sum to four terms; to six terms; to ten terms. In your opinion does the sum of an increasing number of terms increase indefinitely? Consider the accompanying illustration of the sum of an increasing number of terms of this series. Observe that each increment is one half of the remaining distance as we progress toward 2. Can such a sum ever reach 2? Explain the following statements: (1) The sum of n terms of this series can never reach 2, no matter how many terms we take. (2) The difference between 2 and the sum of n terms of this series can be made numerically less than any positive number we may choose, by taking a sufficiently large number of terms. The sum of n terms of this series is said to tend toward 2, that is, $s \rightarrow 2$, as n increases indefinitely. Two is called the *limit of the sum* of n terms of this series. We commonly say that the sum of this unending series is 2. Understand, however, that the series being infinite has no sum in the ordinary sense, and the statement that the sum is 2 must be interpreted to mean that the limit of the sum of n terms is 2 as n increases indefinitely.



Two questions remain to be cleared up: (1) What is the criterion of a limit? (2) How can we recognize a geometric series which has a sum in the sense in which we are using that word? Try to answer both of these questions for yourself.

2. The idea of a limit. We are here using the word "limit" in a special mathematical sense. The criterion of a *limit* in this sense is that the difference between the variable and the constant

which is its limit must become and remain numerically less than any assigned positive quantity. Show that 2 is the limit of the sum of the series $1 + \frac{1}{2} + \frac{1}{4} + \dots$

*3. With the help of the table, find the limit (*a*) of the value of the sine of an acute angle as the angle approaches 90°; (*b*) of the cosine of the same angle; (*c*) of the tangent.

4. Try to find the sum to infinity of the series $1 + \frac{1}{3} + \frac{1}{9} + \dots$ Use the method of Exercise 1 where you studied the series $1 + \frac{1}{2} + \frac{1}{4} + \dots$

5. **How to recognize an infinite geometric progression which has a sum.** Can an infinite G.P. have a sum if $r > 1$? If $r = 1$? Explain and illustrate. When *r is numerically less than* 1, the sum of n terms tends toward a limit as n becomes infinitely large.

6. **The formula for the sum of an infinite geometric series.**

① $s = \dfrac{a - ar^n}{1 - r}$ From Formula II. (Why do we choose this form?)

② $s = \dfrac{a}{1 - r} - \dfrac{ar^n}{1 - r}$ Explain. Consider the value of the second fraction when n is very large and r numerically less than 1.

Show that $s \rightarrow \dfrac{a}{1 - r}$ (See Exercise 2.)

③ **Formula IV.** $s = \dfrac{a}{1 - r}$ We say that the sum to infinity of a geometric progression in which r is numerically less than 1 is $\dfrac{a}{1 - r}$. We mean that this formula gives the limit of the sum of n terms when n increases indefinitely.

Three cautions: In applying this formula remember the condition about the size of r. (What is it?) About n. (What is it?) Remember the special meaning of the word *sum*. (What is it?)

Find the sum to infinity of the series:

7. $1 + \frac{1}{2} + \frac{1}{4} + \dots$

8. $1 + \frac{1}{3} + \frac{1}{9} + \dots$

9. Find on page 141 those geometric series which have "sums to infinity" and find those sums. Such series are called *convergent series.*

Try Exercise 43, A, B, pages 420, 421.

Applications of Infinite Series

Repeating decimals

1. Preliminary study. The fraction $\frac{1}{3} = 0.3333\ldots$; that is, $\frac{1}{3}$ is the sum to infinity of $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots$, or of $.3 + .03 + .003 + \ldots$ Check this conclusion by finding this sum to infinity.

Plan: $a = .3$, $r = .1$, $n = \infty$, (i.e. n is infinite) $s = ?$ Substitute in Formula IV. How does this result check the conclusion above?

Study the fraction $\frac{1}{9}$ in the same way; by expressing it decimally as the sum of an infinite series and by finding the sum of this series.

Such decimals as $0.3333\ldots$, $0.1111\ldots$, $0.2424\ldots$, $0.1313\ldots$, and $.25314314\ldots$ are called *repeating decimals* for obvious reasons. They are sometimes written $0.\dot{3}$, $0.\dot{1}$, $0.\dot{2}\dot{4}$, $0.\dot{1}\dot{3}$, $0.25\dot{3}1\dot{4}$, and so on. Since repeating decimals are infinite geometric series, and since $r = .1$, or $.01$, or $.001$, etc. (i.e., $r < 1$), their values can be found by Formula IV.

2. Find the value of $0.\dot{2}\dot{7}$.

Plan of solution: The G.P. is $.27 + .0027 + .000027 + \ldots$ in which $a = .27$, $r = .01$, $n = \infty$; hence $s = \frac{3}{11}$. (Verify.) Check by expressing $\frac{3}{11}$ as a decimal fraction.

3. Find the value of $.1\dot{2}\dot{4}$.

Plan of solution: $.1\dot{2}\dot{4} = .1 + .024 + .00024 + \ldots$ Notice that $a = .024$. The answer is $0.1 +$ the value of $.0\dot{2}\dot{4}$; that is, $\frac{1}{10} + \frac{?}{?}$. Complete and check.

4. Find the value of each repeating decimal of exercise 1.

Try Exercise 43, C, page 421.

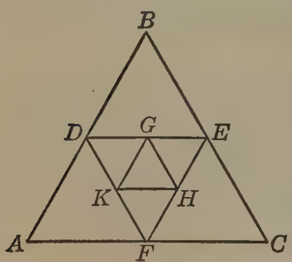
5. Other applications. A rubber ball is thrown to a height of 40 ft. and on each rebound it rises to one half of its previous height. How far does it move before coming to rest?

6. Solve the preceding problem if the words "thrown to" are replaced by the words "dropped from."

7. What distance will a rubber ball travel before coming to rest if it is dropped from a height of 18 ft. and on each rebound rises two thirds of its previous height?

8. Solve the preceding problem if the words "dropped from" are replaced by the words "thrown to."

***9.** If the midpoints of the sides of an equilateral triangle of which the perimeter is 18″ be joined to form a second equilateral triangle, and the midpoints of the sides of this triangle be joined to form a third triangle, and so on, what will the total perimeter of the triangles tend to as their number increases indefinitely?



***10.** If the middle points of a 4″ square are joined in succession so as to form a second square inscribed in the first, and if a third square is similarly inscribed in the second, and so on, what will the sum of the perimeters of all the squares tend to as their number increases indefinitely?

The Progressions. Miscellaneous Exercises

1. Determine which of the following are arithmetic sequences, which geometric, and which neither. For each A.P.

or G.P. supply the next two terms at the right and the next two terms at the left:

(*a*) $3.1, 31, 310\ldots$

(*b*) $3\frac{1}{5}, 1\frac{3}{5}, \frac{4}{5}\ldots$

(*c*) $2\frac{2}{3}, 2\frac{1}{2}, 2\frac{1}{3}\ldots$

(*d*) $1+m, 1+3m, 1+5m\ldots$

(*e*) $-2, -3\frac{1}{2}, -5\ldots$

(*f*) $-4, +\frac{8}{3}, -\frac{16}{9}\ldots$

(*g*) $1, 4, 9\ldots$

(*h*) $1, \frac{1}{2}, \frac{1}{3}\ldots$

(*i*) $\sqrt{3}, 3, 3\sqrt{3}\ldots$

(*j*) $\sqrt{2}, \sqrt{\frac{1}{2}}, \frac{1}{4}\sqrt{\frac{1}{2}}\ldots$

(*k*) $1, 8, 27\ldots$

(*l*) $1, 2, 6, 15, 31\ldots$

(*m*) $1, 3, 6, 10, 15\ldots$

(*n*) $3, \frac{3}{2}, 1, \frac{3}{4}, \frac{3}{5}\ldots$

(*o*) $a^{\frac{1}{3}}, a^{\frac{2}{3}}b, ab^2\ldots$

(*p*) $m^{\frac{1}{4}} - n^{\frac{1}{4}}, -n^{\frac{1}{4}}, -m^{\frac{1}{4}} - n^{\frac{1}{4}}\ldots$

2. Find the sum of ten terms of the sequence of exercise 1 (*a*).

3. Find the sum of ten terms of the sequence of exercise 1 (*e*); of exercise 1 (*i*).

4. Find the sum to infinity of each sequence above which has such a sum.

5. A man has \$1500, and his annual income is \$2800. His annual outgo is \$2400 and is increasing \$50 a year. In how many years will he have \$2250?

6. A clerk receives \$100 a month for his first year, and increases of \$75 per year at the end of each year. What is his salary for the tenth year, and what is his total salary for ten years?

7. A swinging pendulum is gradually coming to rest. The length of the first swing is ten inches and of each succeeding swing 0.9 of the preceding one. What is the length of the eighth swing? What, approximately, is the total distance passed over before the pendulum comes to rest?

***8.** How many ancestors have you had if you are the eighth of your line?

*9. The population of a city is 50,000, and it has been increasing 25% every four years. If it continues this rate of growth, what will be its population at the end of 20 years?

*10. The population of a city is 50,000, and it increases 25% of this number every four years. What population can it expect in 20 years?

*11. The population of Berkeley, California, was 40,000 in 1910 and 55,000 in 1920. If the 10-year rate of increase is arithmetical, what will be the population of this city in 1950? If it is geometrical?

*12. Money invested at 6% interest, compounded annually, doubles itself in 12 years. At this rate what will $100 amount to in 60 years?

*13. A man saves $600 a year out of his salary and invests it regularly in stock paying 7%. If he keeps this up for eight years, what income will he be receiving from the stock at the end of that time? How much money will he have invested?

14. A man buys a phonograph for $400, paying $40 down and $40 a month. The interest on money due is 6%. What will the phonograph actually cost him?

15. Under what condition is $a + b + c$ a geometric progression? An arithmetic progression?

16. Select a value of x so that $(2x - 1) + (3x + 2) + (6x + 8)$ shall be an arithmetic progression.

17. In the sequence 5, 8, 11 . . . which term is 98?

18. How many consecutive terms of the series $96 + 48 + 24 + \ldots$ make a total of $191\frac{1}{4}$?

19. Of how many terms of the series $1\frac{1}{2}$, 3, $4\frac{1}{2} \ldots$ is $1912\frac{1}{2}$ the sum?

20. In the sequence 2, 6, 18 . . ., which term is 486?

21. If $3\frac{1}{3}, 2\frac{1}{2} \ldots$ is an A.P., find the thirtieth term; if a G.P. find the sum to infinity.

22. The tenth term of an A.P. is $\frac{2}{5}$, and the eighteenth term is $3\frac{3}{5}$. Find the series.

23. The seventh term of an A.P. is 17 and the twelfth term is 22; find the first term, the tenth term.

***24.** The third term of an A.P. is a, and the ninth is $6b - 5a$. Find the first and sixth terms and the sum of six terms.

25. Write the eighth term, the fifteenth term, the hundredth term, and the nth term of $2^0, 2^1, 2^2, 2^3, 2^4 \ldots$ Write without exponents the first six terms of this sequence. Write three terms at the left.

26. Write in exponential form with the base 3, six terms of the sequence 81, 27, 9, 3, 1 . . .

27. Write in exponential form at least five terms of a geometric series in which the ratio is 10. Write the same series without exponents.

***28.** A dealer offers an electric refrigerator at $325 on the following terms: $65 down and $20 a month with interest of 7% on the part unpaid. What will the refrigerator cost if bought on these terms?

***29.** A man puts $1000 in a savings bank for his four-year-old son. If the bank pays $4\frac{1}{2}\%$ interest, compounded semi-annually, how much will the boy have at the age of 18 years?

30. Find the sum of $4 + 7 + 10 + \ldots$ to twelve terms? Has it a "sum to infinity"?

31. Find the ninth term of the series $(x - y) + x + (x + y) \ldots$ Find the sum of nine terms.

32. In the series $\frac{1}{9}, \frac{8}{9}, \frac{5}{3} \ldots$ find the eleventh term and the sum of the first eleven terms.

33. Find the sum of 8 terms of the series $5 - 3\frac{1}{3} + 2\frac{2}{9} - \ldots$

34. How many terms are there in the series

$$-\tfrac{14}{3} - \tfrac{11}{3} - \tfrac{8}{3} - \ldots \text{ if the sum is } -\tfrac{28}{3}?$$

***35.** A farmer sowed $\frac{1}{8}$ bushel of corn and used the whole crop for seed the following year, and so on to the fourth crop, which amounted to 8192 bushels. If r was constant, what was the amount of the second crop?

***36.** A seed grower sowed one quart of seed and harvested a crop of one peck. If he continues to plant his whole crop each year and r remains constant, in how many years will his crop first exceed 1000 bushels?

***37.** A man deposited $2500 in the savings bank on the day his son was born. With interest at 4% compounded semi-annually, how much should the son receive at the end of his 21st year?

***38.** If $200 is deposited at the beginning of each year for 10 years and interest is 4% compounded annually, what will be the amount at the end of the tenth year?

39. In the progression $8 - 12 + 18 - \ldots$ write the 11th term and the sum of 30 terms without computing the values.

***40.** A $1500 debt on an automobile is to be paid off in monthly installments of $50 and interest at 12%. What is the total amount to be paid?

***41.** Insert 12 geometric means between 1 and 2. (This is the problem of dividing an octave into 12 equal semitones as is done on a piano, for example.)

Reviews and Tests

Test A

1. Define and illustrate an arithmetic progression; a geometric progression.

2. Derive three formulas for arithmetic progressions.

3. Derive four formulas for geometric progressions.

4. What four steps in an algebraic study are suggested in the introduction to this chapter?

***5.** A book on biology says, "In one progression each change is proportional to the original value; in the other progression each change is proportional to the value at the beginning of the interval." Comment on this definition.

6. Write in literal symbols a general arithmetic sequence; a general geometric sequence.

Test B

1. Find the sum of eight terms of $8 + 6 + 4 + \dots$

2. Find the sixth term of $12 - 6 + 3 - \dots$

3. Find the arithmetic mean and the geometric mean between 300 and 1200.

4. Select the series below which has a sum to infinity and find that sum:

$$5 + 4 + 3 + \dots$$
$$5 + 4 + \tfrac{16}{5} + \dots$$
$$5 + 4 - \tfrac{16}{5} + \dots$$

5. Insert two geometric means and two arithmetic means between 6 and 48.

Test C

1. If the sequence $2\frac{1}{2}$, $3\frac{1}{3}$, is arithmetic, find the twentieth term; if it is geometric, find the fifth term.

2. Find the value of .916888...

3. Given a, n, and s in an A.P. derive a formula for d.

4. From the formulas for G.P. derive a formula for a in terms of l, r, and s.

5. Find the sum of all the multiples of 3 between 100 and 275.

6. What is the common fraction equivalent to $.3\dot{1}3131\ldots$?

*7. To what amount will \$125 grow if placed at interest at 6% compounded annually for 100 years?

8. A balcony is to contain 828 seats. The first row has room for 58 seats, and each succeeding row for 2 seats more than the row in front of it. How many rows will be required?

*9. Insert two arithmetic means between 1 and 5. Insert two geometric means between the same two numbers.

10. Find to three-figure accuracy the value of five terms of $1 + \frac{1}{3} + \frac{1}{9} + \ldots$ What is the difference between this sum and the sum to infinity?

*11. If the price in cents of a lottery ticket is the number of the ticket, and the tickets are numbered consecutively from 1, how many tickets must be sold in order to raise \$18.30? *What is the smallest number which can be sold in order to take in not less than \$50?

Test D

Questions similar to those set for college entrance.

1. Solve for x and y the simultaneous equations:

$$\frac{a}{b+y} = \frac{b}{a-x} \text{ and } \frac{a}{b+x} = \frac{b}{a-y}$$

2. Solve for x: $\dfrac{x^2}{c^2} = \dfrac{ax+b}{ac+b}$

3. Compute the value of $\dfrac{.08796\sqrt{0.3510 \cos 38.41^\circ}}{\sin 76.34^\circ}$

4. Give the meaning of each of the following laws of exponents. State in words or symbols four corresponding laws for logarithms.

I. $x^a \cdot x^b = x^{a+b}$ II. $x^a \div x^b = x^{a-b}$

III. $(x^a)^n = x^{an}$ IV. $\sqrt[r]{x^a} = x^{\frac{a}{r}}$

5. Write in logarithmic form $10^{-2} = 0.01$, $10^{2.8727} = 746$.

6. By taking the logarithm of each number solve the equation $10^x = 24.93$.

7. Multiply $[(a + b) + 2]\,[(a + b) - 2]$.

8. Factor:

$\pi r_1^2 - \pi r_2^2$ $a^2 - 22\,a + 120$ $16\,a^4b^2 - 88\,a^2bc^3 + 121\,c^6$

9. Find the base angles of an isosceles triangle of which the base is 56.44′ and the equal sides each 72.31′. Use logarithms.

10. If $x = 4\,y$ and $5\,x = x^2 - y^2$, substitute the first equation in the second and find the values of x and y.

***11.** Show that the division of $1 - r^n$ by $1 - r$ gives a geometric progression.

In Preparation for Chapter V

Repeat **Exercises 18, 19, C, 20, 21, F,** pages 363–376.

CHAPTER V

PROBLEM SOLUTION †

The scientific method. The present age is preëminently scientific. It is set apart from all other ages by scientific discoveries and inventions which are the result of a scientific method of thought and a scientific attitude of mind. The method of science is briefly this: to study a situation, to find the data connected with it, to set these data in order, to analyze them, to discover what conclusions follow necessarily from them, and what interpretations are consistent with them. The analysis and solution of verbal problems gives one of the best opportunities to become familiar with the scientific method and to secure practice in its use. The important subject matter of algebra problems lies largely in the other sciences, but since these sciences are unfamiliar to many students of algebra and cannot very well be taught in the algebra class, it is often necessary to make use of fictitious problems and puzzle situations in order to develop the problem-solving ability.

Overcoming the difficulties. The solution of a verbal problem is accomplished by a series of logical steps and usually no one of the steps is in itself difficult. If a problem appears long and involved on first reading, do not think, "I cannot do it," but think, "I can take the first step, etc." When in difficulty use your ingenuity and refer as often as necessary to the *plan for problem solution* given on the following pages. It is interesting to notice that when you have derived an equation from a problem, the solving of this equation is reasoning about the numbers

† This chapter may be studied as a whole or in part at any time after Chapter I has been completed. The ability to solve verbal problems sometimes develops very slowly, and many classes find it advantageous to solve one or more of such problems nearly every day throughout the course. The growth of ability to solve verbal problems is probably our best test of the effectiveness of algebra as a means of education.

of the problem. The reasoning is systematic and, if it obeys the laws for the use of algebraic symbols, is certain to be correct.

A Plan for Problem Solution

On pages 170, 171 there are listed six steps to take in solving verbal problems.

Thoroughly master these steps, and refer to them whenever necessary. They will help you, not only in solving verbal problems, but also in forming correct habits of thought.

The six steps are briefly summed up in

The three fundamental habits to form

First: See what numbers are involved in the problem.

Second: Find the relations between these numbers.

Third: State these relations in algebraic symbols.

After you have thought over these three habits and after you have mastered the six steps on the two pages following, study the somewhat involved problem below:

At the first of two games a reserved seat cost $1.50, a bleacher seat cost 50 cents, and the number of reserved seats sold was $\frac{1}{25}$ of the number of bleacher seats sold. At the second game the reserved seats are to be reduced to $1.00 each and the same total number of seats is to be sold. What part of them should be reserved in order to give the same receipts?

The numbers involved are indicated by the top and side headings below: the relations may be represented as shown. The relation wanted will then be $\frac{y}{26x} = ?$ An equation may be based upon the equality of the receipts at the two games.

	No. of R. seats	No. of B. seats	Cost of R. seats	Cost of B. seats
1st game	x	$25x$	$150x$	$1250x$
2nd game	y	$26x - y$	$100y$	$50(26x - y)$

The answer is $\frac{1}{13}$.

Try Exercise 44, A, page 422.

The steps to take

I. *Read the problem* thoughtfully, phrase by phrase. As you read, do not try to discover the answer, but to "understand the situation" and

A. Try to discover all the numbers involved. They are described or implied in the phrases of the problem.

B. Try to find relations between these numbers. These relations are

Stated in words, or *implied* in such words as rate, distance, sum, perimeter, right triangle, interest, cost, etc.

(Remember that the development of the ability to discover and make use of the relations between numbers is one of the fundamental aims of this course. Failure to recognize an implied relation is a common cause for failure in problem solution.)

C. Try to select (a) the plan of solution (A, B, or C under II below) and also (b) the relation upon which the equation, or the equations, will depend.

II. *List the unknown numbers* of the problem. Either

A. Write a verbal description of each unknown number. (This plan is illustrated in Step I of Prob. 4, page 44.)

B. Draw a sketch and label its parts. (This plan is illustrated in Problem 47, page 50.) Or

C. Use a tabular arrangement. (This method is illustrated in Problems 4 and 7, page 45, and Problem 37, page 49; it saves time and aids thinking; it is nearly always helpful in problems in which the relation between the numbers is in the form $a = rb$.)

(In listing the unknowns, study the words and phrases of the problem which describe numbers.)

III. *Assign an algebraic symbol* to each unknown number in the list.

A. Assign one letter to the unknown in terms of which the others can most easily be represented. Often this unknown will be the answer called for by the problem.

B. If there is a second unknown which does not appear to be closely related to the others, assign to it a second letter.

C. Assign an expression in this letter or in these letters to each of the other unknowns in the list. In taking this step, study carefully the words of the problem which describe or imply relations between numbers. As you proceed, select, but do not use in the list, the relation upon which to base the equation, and select the units in which the equation will be expressed.

IV. *Form one or more equations.*

A. By substituting in a formula (see Prob. 47, page 50), or

B. By equating two representations of the same number, or

C. By writing in algebraic symbols some equality stated or implied in the problem. Use for this purpose some relation not already used in Step III.

(Make sure that both members of the equation are expressed in the same units.)

V. *Solve the equation or equations.* Use a systematic method of procedure which any mathematician can read with understanding.

VI. *Check the result* in all the conditions stated in the problem. Give attention to:

A. Any results which will not check.

B. Negative results, which may need to be rejected or especially interpreted.

C. Irrational results and the degree of accuracy to which they are to be expressed.

D. Imaginary results. (See page 40.)

The Digit Problem

It is not a bad idea to think a little about the fundamental principle upon which our decimal number notation is based. What is the meaning of the 8 and the 4 in 843? Of the zero in 207? In the following problems such relations are involved.

1. In a two-digit number the sum of the digits is six. If the order of the digits is reversed, the resulting number will be 18 greater than the original number. Find the number.

Plan of solution: The implied relation is: the number is ten times the tens digit plus the units digit, that is, $10\,t + u$. (Explain and illustrate.)

II, A.† = the units digit
= the tens digit
= the number
= the number with digits interchanged

Show that these numbers could conveniently be arranged in a table.

IV, C. ① $10\,u + t = 10\,t + u + 18$

For a somewhat more practical use of the principle here employed, see *Everyday Algebra*, page 285.

2. A number is expressed by two digits of which the sum is 12. If the digits are interchanged, the resulting number is 36 less than the original number. Find the number.

3. The sum of the digits of a two-digit number is 8. If six times the tens digit is added to the number, the digits will be interchanged. Find the number.

4. A certain number is equal to eight times the sum of its two digits. The first digit exceeds the second by five. Find the number.

5. The sum of the two digits of a number is four times the tens digit. If the digits are interchanged, the resulting number exceeds the original number by 36. Find the number.

† The Roman numerals refer to the scheme on pages 170, 171.

6. The tens digit of a number is three times the units digit. If the number is divided by the sum of its two digits, the quotient is 7 and the remainder 6. Find the number.

7. The sum of the digits of a two-digit number is 9. If 1 be added to each digit and then the digits be interchanged, the new number is 2 more than 3 times the original. Find the number.

8. Twice a certain two-digit number is equal to one more than the number with its digits reversed; and the units digit is one more than twice the tens digit. Find the number.

***9.** The sum of the digits of a three-digit number is 11, and the hundreds digit is half the tens digit. If the digits be reversed, the new number is greater than twice the original number by a two-digit number made up of the units and hundreds digit respectively of that original number. Find the number.

Problems Based upon the Pythagorean Relation $a^2 + b^2 = c^2$

1. The lengths of the sides of a right triangle are given by three consecutive even numbers. Find them. (Plan II, B)

2. Construct a right triangle in which the hypotenuse is eight feet longer than the shortest side and the third side seven feet longer than the shortest side. Find the sides.

3. Construct a right triangle in which the hypotenuse is one inch more than twice the shortest side, and the third side is one inch less than double the shortest side. Find the sides.

4. The sum of the perpendicular sides of a right triangle is 35 ft., and the hypotenuse is 25 ft. Find the sides.

5. In the middle of a pond ten feet square grew a reed. The reed projected one foot above the surface of the water. When blown aside by the wind without bending, its top reached the midpoint of the side at the surface of the water. How deep was the pond? (Old Chinese problem.)

6. Forty-six rods of fencing will just surround a rectangular field of which the diagonal is 17 rods. What are the dimensions of the field?

7. The diagonal of a square is five feet longer than a side. Find the side to three significant figures.

8. The sides of a right triangle are such that one leg is 7 inches longer than the other, and the hypotenuse is 25 inches longer than the same leg. Find the sides.

9. The difference between the base and the hypotenuse of a right triangle is 1 inch, and the hypotenuse is 5 less than six times the third side. Find the dimensions.

***10.** Show that $(a^2 + b^2)^2 - (a^2 - b^2)^2 \equiv (2\,ab)^2$. If $a = 5$ and $b = 4$, find the value of each of the preceding expressions. Show that the square roots of these values may be used as length numbers of the sides of a right triangle. With the help of the formula find other sets of such numbers. Justify the use of the formula for this purpose.

Problems in Which the Sum or Total is Known

1. How large a field can be surrounded with 920 rods of fencing if its width must be 20 rods less than half its length?

Plan of solution: Draw a sketch in order to help visualize the situation and to save time in describing the quantities which are represented algebraically. Plan II, B. Equate two values of the length of the perimeter.

2. Solve the preceding problem if the ratio of the length to twice the width is 1.8.

3. Solve problem 1 if the perimeter is 520 feet and the difference between length and width 200 feet less than the sum of length and width.

4. The perimeter of an isosceles triangle is 220 in. and each of the two equal sides is 10 in. more than twice the base. Find the lengths of the sides.

5. The perimeter of a rectangular field with a 25 rod diagonal is 62 rods. Find its length and width.

6. The dimensions of a rectangular box are expressed in inches by three consecutive integers, and the surface of the box, including the top, is 292 square inches. What are the dimensions of the box?

7. In a machine two forces are to act differently at different times, sometimes both acting in the same direction and sometimes one opposing the other. When acting together they make a total of 30 tons; when opposed, their combined effect is 4 tons. Find the forces.

8. Solve the preceding problem if the forces acting together total 20 tons, and opposed 16.6 tons.

9. A man flies a mail plane 140 miles with the wind in 1 hour, and 37 miles against the wind in half an hour. Find the rate of the wind.

***10.** When the sum of the first three terms of an A. P. is half the sum of the first seven terms and greater by 14 than the sum of the second group of seven terms, what is the 16th term?

Problems in Which $a = rb$

In each problem below, one of the numbers involved is the product of the other two numbers. In each solution the *tabular arrangement of data* (II, C) will help in finding the numbers, describing them, recognizing their relations, and hence in representing them algebraically. Remember that the habit of arranging data in tables is one of the valuable habits to acquire in the mathematics class.

Re-read pages 169–171. Study the method of the following model solutions until you understand them thoroughly.

1. Two young men with automobiles agreed to meet at a certain point at a certain time. The first had 27 miles to go, the

second 25. The second decided to start 12 minutes earlier than the first and to drive $2\frac{1}{2}$ miles an hour slower than the first. If both are to arrive at the same time, how fast must each drive?

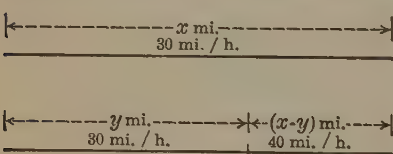
Plan of solution (II, C):

	$d =$	r $\times$	t
First man	27	x	$\frac{27}{?}$
Second man	25	$x - 2\frac{1}{2}$	$\frac{25}{?}$

Observe that whenever two spaces in a horizontal row are filled, the third may be filled with the help of the relation $d = rt$, and without reference to the statement of the problem. Comment on the units used in the problem.

IV, C. The time of the first equals the time of the second less 12 minutes. (It is a good plan to state the equation in words and then to translate it into algebraic symbols.)

2. A similar but somewhat more complicated puzzle problem taken from an old examination paper reads: Two automobiles race, starting together. The first runs at a uniform speed of 30 miles an hour. The second runs at a uniform speed of 35 miles an hour until it is held up by a breakdown which delays it for one hour. Starting again at a uniform speed of 40 miles an hour, it overtakes the first automobile in one hour. How far from the start does this overtaking occur? *Plan of solution*, II, B:



II, C.		$d =$	r $\times$	t
First		x	30	$\frac{x}{30}$
Second	(a)	y	35	$\frac{y}{?}$
	(b)	$x - y$	40	$\frac{x-y}{?}$

IV, C. Time of the first − 1 = total time of the second. Time of the second (b) = 1 hour. Complete and check.

3. An airplane starts from B at 60 miles an hour at the same time another plane traveling three times as fast starts to overtake it from A 50 miles behind. When will they first be 20 miles apart?

Plan of solution (II, B):

II, C.	d	$=$	r	$\times t$
First	60?		60	x
Second	180?		180	x

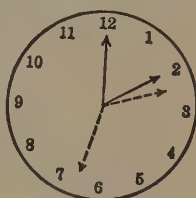
IV, C. The first distance + 50 = the second + 20. Explain with the help of your drawing.

Form an equation, solve, and check. In checking make use of the drawing. If the airplanes continue in the same direction at the same speeds, when will they again be 20 miles apart?

***4.** How many minutes after 2 o'clock will the hands of a clock be 20 minute spaces apart for the first time?

Plan of solution, II, B:

II, C. Follow the method of the preceding problem. Make use of the implied relation that one hand travels 12 times as fast as the other. (The rate of the minute hand is 1 space per minute.)



IV, C. The distance for the long hand = the distance for the short hand + 10 + 20. Explain with the help of the sketch.

Show that the answer is $32\frac{8}{11}$ minutes. Check it.

5. A boy and his father are going to spend their vacation in the country. The boy starts out in the morning and walks until lunch time, when he has covered fifteen miles. At two o'clock he sets out again and walks on at an average of $2\frac{1}{2}$ miles

an hour. At the same time his father starts from home in his automobile at a speed 12 times as great. When will the father overtake the boy? (Carefully compare the solutions of problems 4 and 5.)

6. At what time after 3 o'clock will the hands of a clock be together?

7. Frank can plow a field in ten days and his father can plow it in 5. How long will it take them to plow it if they work together?

Plan of solution: The amount done equals the rate times the time. The rates are known. Since Frank can plow the field in 10 days, his rate is $\frac{1}{10}$ of the field per day. What is his father's rate?

II, C.	$a =$	$r \times t$		
Frank	$\frac{x}{10}$	$\frac{1}{10}$	x	In what order should these items be written in the table?
Father	$\frac{x}{5}$	$\frac{1}{5}$	x	

IV, C. The relation which may be used for the formation of the equation is that the part Frank does + the part his father does = 1; that is, it equals the unit of work completed. (Give other illustrations that the sum of all the fractional parts of a thing equals 1.) Complete and check.

8. John can do a piece of work in $4\frac{1}{2}$ days. After he has worked at it 3 days, Will comes to help him, and they finish in $\frac{7}{8}$ of a day. Will wants to know the number of days in which he alone could have done the work.

Plan of solution:	$a =$	$r \times t$	
John (alone)	$1 =$	$\frac{1}{4\frac{1}{2}}$	$4\frac{1}{2}$
Will (alone)	1	$\frac{1}{x}$	x

Together	John	$\frac{3\frac{7}{8}}{4\frac{1}{2}}$	$\frac{1}{4\frac{1}{2}}$	$3\frac{7}{8}$
	Will	$\frac{\frac{7}{8}}{x}$	$\frac{1}{x}$	$\frac{7}{8}$

IV, C. The part John does in $3\frac{7}{8}$ days + the part Will does in $\frac{7}{8}$ day = 1. Complete and check.

9. A contractor must remove 2000 tons of rock in 22 days. His steam shovel can remove 62 tons a day. For \$50 a day he can rent another shovel which will remove 95 tons a day. The excavation is narrow and only one shovel can be used at once. What is the smallest rent he will have to pay? (He cannot rent the shovel for part of a day.)

Plan of solution: (The cost is asked for in order to complicate the problem by somewhat disguising the fact that the number to seek is the number of days the second shovel is to work.)

	a	$= r \times$	t
First shovel	$62(22 - x)$	62	$22 - x$
Second shovel	$95x$	95	x
Total	2000		

In what order should these items be written in the table? Which vertical column contains items which suggest that they may be used in the equation?

IV, C. The total number of tons excavated by both shovels is 2000.

10. A dairyman is mixing 3% milk with 5% milk in order to obtain 20 gallons of 4.5% milk. How much of each kind should he use? (The per cents refer to per cents of butter fat contained in the milk.)

Plan of solution: II, C. The total amount of butter fat equals the per cent times the amount of milk.

	$b =$	p	$\times m$
3% milk	?	.03	x
5% milk	?	.05	$20 - x$
Total	?	.045	20

IV, B. Observe that when the first column is completed, you have two methods of representing the total butter fat. Complete and check.

11. How many pounds of 4% salt solution must be mixed with 24 pounds of a 12% solution in order to make a 10% solution?

Plan of solution:

	$s =$	$p \times$	S
12% sol.	?	.12	24
4% sol.	?	.04	x
10% sol.	?	.10	$24 + x$

IV, B. See the comment IV, B in the preceding problem.

12. How much water must be added to 8 ounces of an iodine solution one-quarter pure to secure a new solution 10% pure?

13. It is desired to divide a 90 ft. strip of motion picture film into individual pictures of such width that if each is made $\frac{1}{4}''$ wider there will be 216 pictures less. What width should be used? (For the headings of the table use: Width of a picture times number of pictures equals length of film. Notice also the units used.)

Repeat Exercise 44, A, page 422.

14. If you had 10 hours at your disposal, how far could you walk out into the country at 3 miles an hour if you knew that you could get a ride back at the rate of 20 miles an hour? (After you have found the answer and expressed it as a mixed number, tell whether you consider this a practical method of expressing such a distance. Contrast it with distances measured on a speedometer.)

15. On account of fog the usual speed of a train was decreased 4 miles an hour. The 180-mile trip took half an hour longer than usual. What was the usual speed of the train?

16. On a recent crossing from Southampton to New York bad weather compelled the Steamship Aquitania to reduce her normal average speed from 24 knots to 9 knots during part of the voyage; and because of this, the crossing was extended from $5\frac{1}{2}$ days to $7\frac{1}{2}$ days. For how much of the time did the liner travel at 9 knots? (A knot means one nautical mile an hour.)

17. A truck driver agreed to drive 105 miles in a certain time. After driving 63 miles at a uniform rate which would just enable him to keep his agreement, he was delayed 24 minutes. He increased his speed $3\frac{1}{2}$ miles an hour and arrived exactly on time. What was his original rate?

18. A taxi driver agreed to drive 60 miles in a certain time. After driving 36 miles at a uniform rate which would just enable him to keep his agreement, he was delayed 15 minutes. He increased his speed 8 miles an hour and arrived exactly on time. What was his original rate?

***19.** An expert typist had 6000 words of copy to complete in a certain time. She finished one third of it on time and was then called away from her work for half an hour. In order to complete her task, in the specified time, she was forced to do the remainder of the copy at a speed ten words a minute faster than that at which she wrote the first part. What was the time required for the typing?

20. A man rows downstream a distance of 15 miles and back again in 6 hours. He finds that he can row 5 miles with the stream in the same time as 3 against it. Find his rate of rowing in still water. (Notice that the boat has two rates; that in still water plus the rate of the current, and that in still water minus the rate of the current.)

21. A man rowing on a river which flows two miles an hour finds that it takes him three times as long to row a mile upstream as to row a mile downstream. Find his rate of rowing in still water.

22. A bird flying with the wind goes 55 miles an hour and flying against a wind of twice the velocity, 30 miles an hour. What are the rates of the wind?

23. An observer notices that the velocity of a gunshot with the wind is 1125 feet a second and against the wind 1045 feet a second. What is the rate of the wind? Express the answer in miles per hour.

***24.** In a race of 440 yards, George gave Harry a start of 20 yards and beat him by two seconds. In a second race George gave him a start of 4 seconds and beat him by 6 yards. What are the rates of the boys?

25. A does $\frac{3}{5}$ of a piece of work in 10 days; then he calls in B to help him, and they finish the work in 3 days. B wants to know how many days it would have taken him to do the work by himself.

26. A painter can paint a house alone in 5 days, and an apprentice can do it alone in 15 days. In how many days can they do it if they work together?

27. A tank can be filled by two pipes in 24 minutes and 30 minutes respectively, and emptied by a third in 20 minutes. In what time will it be filled if all three pipes are open?

28. If A can do a piece of work in 8 days and A and B together in 6 days, how long will it take B to do it alone?

29. A can do a piece of work in 6 days and B can do it in 14 days. A began the work but abandoned it, and B took it up immediately and completed it in a total of 10 days. How many days did A work?

30. One pipe can fill a certain tank in 24 hours and another pipe can empty it in 19 hours. If the first pipe is opened into an empty tank at 8 A.M. and the second is opened at 11 A.M., at what time will the tank be empty?

31. How much rice at 8 cents a pound must be mixed with 20 pounds at 11 cents a pound in order that the mixture may be worth 10 cents a pound? (For the headings of the table use, no. of lb. $\times$ cost a lb. = total cost.)

32. How many ounces of another metal must be added to 56 ounces of pure silver in order to make a composition which shall be 70% silver?

33. How many pounds of fresh water must be added to 32 lb. of sea water which contains 16% salt by weight in order to reduce the salt to 2%?

34. Each of two bins contains a mixture of corn and oats. The first has 22% corn, the second 42% corn. How much must be taken from each bin to make a mixture of 40 bushels containing 30% corn?

35. A certain heavy syrup contains 20% of water, and another contains 28% of water. How many gallons of each must be used in order to obtain 50 gallons of a syrup containing 21.6% water?

36. A photographer has two bottles of diluted developer. In one bottle 10% of the contents is developer and the rest water. In the other the mixture is half and half. How much must he draw from each bottle to secure 8 ounces of developer which shall be 25% pure?

37. It is believed that the rectangle most pleasing to the human eye is one of which the sum of the two dimensions is to the longer as the longer is to the shorter. A page with a perimeter of 20 inches is made up according to this principle. What are its dimensions?

38. A rectangular field is 20 rods longer than it is wide. The area is 2400 square rods. Find the dimensions of the field.

39. The length of a rectangular field is 4 rods more than the width, and the area is 117 sq. rods. What are the dimensions of the field?

40. A rectangular sheet of paper twice as long as a square piece and 3 in. wider than the square piece contains 108 sq. in. Find the length of the square piece.

41. Two small flower beds have equal areas. One, which is rectangular, is 3 ft. longer and 2 ft. narrower than the other, which is square. Find the dimensions of each bed.

42. A man has \$3500 at interest, part at $3\frac{1}{2}\%$, the rest at 4%. His total income from the investment is \$130 a year. How is the money divided? For the headings of the table use:

Principal $\times$ rate = income.

43. A man has \$3000 invested, part at 5% and the rest at 6%. His total income is \$157 a year. How much is invested at each rate?

44. A boy made two investments totaling \$710. On the first he received $5\frac{1}{2}\%$ interest, and on the second 4%. His income for the year was 10 cents more on the first investment than on the second. What were the sums invested?

***45.** A woman borrowed a sum of money and paid 5% interest on it for 12 years, when she found that the amount of interest paid was only \$240 less than the sum borrowed. How much did she borrow?

46. Ten thousand dollars is invested, part at 6% and the rest at 4%. The simple interest on the first part for four years is the same as that on the second part for two years. Find each part.

47. In a certain payroll there were twice as many two-dollar bills as there were fives, and as many ones as twos and fives together. The payroll amounted to \$25,200. How many bills of each denomination were there?

Try Exercise 44, B, page 422.

Miscellaneous Problems

1. A truck took a heavy load to a destination 72 miles distant and returned empty. The average speed going out was 12 miles an hour less than the average speed on the return trip, and the total driving time was 9 hours. Find the speeds.

2. A boy is to start from home and walk at the rate of $3\frac{1}{2}$ miles an hour; $2\frac{1}{2}$ hours later his brother is to start from the same place on his bicycle and ride after him at the rate of $8\frac{1}{2}$ miles an hour. In how many hours will the rider overtake the pedestrian?

3. A man has \$8000 which he wishes to invest in two enterprises, one of which is to pay $5\frac{1}{2}\%$ and the other 5%. How much must he invest in each in order to receive \$425 from them both?

4. The length of a rectangular field is twice its width. It cost as much to fence it at 50 cents a yard as to sod it at 15 cents a square yard. Find its dimensions.

5. A picture 8 inches by 12 inches is surrounded by a frame of uniform width. The area of the frame equals the area of the picture. What is the width of the frame? (Plan II, B.)

6. A picture twice as long as wide is surrounded by a frame 4″ wide. The area of the frame is 328 inches. What are the dimensions of the picture?

7. A and B can do a piece of work in six days. A alone can do it in 10 days. How long will it take B alone to do it?

8. A and B can do a piece of work together in 4 days. A works alone for three days and then is joined by B, and the two men complete the work in 3 days more. In what time could each do the work alone?

9. A small motor boat which can go 8 miles an hour is to make a trip downstream and back on a river which flows 2 miles an hour. How far downstream can the boat go if 6 hours is available for the round trip?

10. A father engaged his son to work on the condition that the son was to receive \$2 for each day that he worked and forfeit \$1 for each day that he was idle. At the end of 20 days \$34 was due the son. How many days had he worked?

11. Many great paintings have their main figures so placed that the distance from one edge of the picture to the center of the main figure is the mean proportional between the remaining width and the entire width of the picture. This point is how far from one side of a picture 16 ft. wide? (To nearest tenth.)

12. Answer the question of example 11 for a picture 38″ wide.

13. A rectangle is of dimensions pleasing to the human eye if the width is to the length as the length is to the sum of the two dimensions. A page designed on this plan is to have a perimeter of 24 inches. Find its width. Answer to two-figure accuracy.

14. A line 2 in. long is to be divided into two parts so that the longer segment is a mean proportional between the shorter segment and the whole line. Find to three-figure accuracy the length of each segment.

15. Set a selling price for a piano costing \$218 so as to allow for overhead 40% and for profit 18% of the selling price.

Selling Price (s)

Cost \$218	Overhead $.40s$	Profit $.18s$

Plan of solution:

① $218 + .40\,s + ? = s$

16. The cost of manufacturing a certain automobile is \$900. Set a selling price for this car so as to allow 30% of the selling price for overhead and 15% of the selling price for profit.

17. A radio set cost a merchant \$72. At what price should he mark the set so that he can allow a deduction of 10% from the marked price and still make a gross profit of 20% of the marked price?

18. A dealer pays \$8.25 for a ton of anthracite coal at the mine. Set a selling price so as to allow 25% of the selling price for profit and 20% for overhead. (Assume that ton means short ton in each case.)

***19.** A firm promised its manager a bonus of 10% of its net profit after the 5% income tax had been paid. Upon the bonus no income tax need be paid. The net profit was \$20,000. What was the tax? What was the bonus?

20. One man has \$52 and is saving at the rate of \$2 a day. Another man has \$325 and is spending it at the rate of \$1 a day. In how many days will they both have the same amount?

21. Two men buy together 500 lb. of grain, A taking 60 lb. more than B. A uses it at the rate of 15 lb. a day and B at the rate of 11 lb. When will they both have the same amount left?

22. A motion was passed by a majority of 64 votes. On reconsideration later, one ninth of those voting for it changed their votes and it was defeated by a majority of 16 votes. From these data can you work back to the number who voted for the motion originally? How many voters were there?

23. A certain number of pupils took an algebra test, and four more than three times as many passed as failed. On a second test a week later four who had passed the first test failed and seven who had failed the first test passed. $16\frac{2}{3}\%$ of the class were then failing. How many took the tests?

24. A grocer is mixing 40-cent coffee with 60-cent coffee to make a mixture of 70 lb. worth 54 cents a pound. How many pounds of each kind should he put in?

25. How much tea costing 54 cents a pound must be mixed with tea costing 75 cents a pound to make 18 pounds of a mixture that will cost 68 cents a pound?

26. A man buys coal at $16 and at $12 a ton. If he buys 10 tons for $140, how many tons does he buy at each price?

***27.** A grocer wishes to make a "Special" table of canned goods to sell at "3 cans for 64 cents this week only." He puts on the table 54 cans of peas and corn, the first selling normally at 24 cents a can, the latter at 18 cents a can. How many of each are there?

***28.** A grocer has 21 boxes of cookies that sell at 8 cents a box and 17 boxes that sell at 12 cents a box. How many boxes selling at 14 cents each must he add if he is to advertise, "Any two boxes for 23 cents"?

29. Two aviators starting from the same point at the same time are traveling one due north and the other due west. The second travels 70 miles per hour faster than the first. How fast must each travel if they are to be 520 miles apart at the end of 4 hours?

30. Try to solve the preceding problem if the aviators at the end of 4 hours are to be 500 miles apart. 40 miles apart. Discuss the results.

31. Two steamships leave the same place and travel one due east at 13.75 miles an hour, the other due north at 12 miles an hour. In how many hours will they be 73 miles apart?

***32.** The lengths of two parallel chords in a circle are 6″ and 8″ respectively, and the distance between them is 1″. Find the radius of the circle.

33. The top of a tree 64 feet tall is cut partly off and tips over so that the top touches the ground 48 ft. from the base of the stump. Find the height at which the cut is made.

34. Two machines working together coaled a ship in 6 hours and 24 min. At another time one of the machines coaled the ship in 10 hours, 40 min. The next time the other machine only will be available. How long must the captain allow for coaling?

35. At what price should a dealer mark a piano costing him \$156 so that he can allow a discount of 10% of the marked price and still make a profit of 20% on his investment?

36. What price should he put on a desk costing \$72 so that there may be a profit of 20% of the selling price after 15% of the selling price has been used to pay expenses?

37. A bar of copper and zinc weighing 120 pounds is to be sold for \$9.90. If copper sells at 12 cents and zinc at 7 cents a pound, how much of each metal must be used?

38. A girl paid \$4.80 for nuts and sold them at a profit of 10 cents a pound. With the proceeds she was able to buy 12 pounds more than she bought the first time. How many pounds did she buy at first?

39. A man had \$12,300 invested in two enterprises. On checking up at the end of the year he discovered that one had paid him $7\frac{1}{2}$% and that on the other he had lost $3\frac{1}{4}$%. His net profit for the year was \$62.50. How much money had he invested in each enterprise?

40. A man has \$5500 invested in two enterprises, one paying 5% and the other 6%. His annual income is \$304. How much of the first investment must he change over to the second to secure an even \$320 a year?

***41.** The formula $F = k + k_1w$ gives the relation between the force applied to a certain machine and the weight lifted. k and k_1 are constants depending upon the construction of the machine. If a force of 6 pounds will raise a weight of 6 lb., and a force of 14 lb. will raise a weight of 30 lb., find k and k_1 and then find the force necessary to raise 60 lb., and the weight which will be raised by a force of 10 lb.

***42.** A and B run one mile (1760 yards). First A gives B a start of 44 yards and beats him by 51 seconds. Next A gives B

a start of 1 minute and 15 seconds and is beaten by 88 yards. How long does it take each to run the mile?

*43. A man and wife and child traveled from Boston, Massachusetts, to Burlington, Vermont. The mother and child shared the lower berth and paid $16.38 for fare and berth, the child traveling half fare. The father occupied the upper berth and paid $11.42 for fare and berth. The lower berth cost $.75 more than the upper. What was one full fare?

*44. A girl's grades at the end of two terms averaged 80%. What must she make for the third term to average 85% for the year?

45. A boy has an average of 45 on three tests. There are only three more before his grades are sent in. If he gets 70 on one of these, what is the lowest grade he can get on the other two and still have a passing average of 55?

*46. A team that has played 38 games has a standing of .711. There are 14 more games to be played. How many must the team win in order to end the season with a standing of .750?

47. Find the dimensions to be used in making a cylindrical can which shall hold 27π cu. in. and in which the lateral area shall equal the combined area of the top and bottom. (The lateral area equals $2\,\pi r h$ and the volume equals $\pi r^2 h$.)

*48. What portion of a sphere is cut off by a plane cutting the sphere half way between the center and the surface? V (of sphere) $= \frac{4}{3}\,\pi r^3$; V (of segment) $= \frac{1}{3}\,\pi h^2\,(3\,r - h)$.

49. A small boy can buy marbles at 3 for 5 cents. If he can sell $\frac{1}{3}$ of them at $1\frac{1}{2}$ cents each and the rest at 2 cents each, how many must he buy in order to make a profit of $1?

50. Find an arithmetic progression of which the first term is 7 and of which the second, fifth, and tenth terms form a geometric progression.

***51.** A and B run a race of one mile (1760 yards). A gives B 12 seconds start and beats him by 44 yards; then A gives B 165 yards start and is beaten by 10 seconds. Find their rates.

***52.** A traveler started on a journey of 330 miles, leaving 4 hours and 12 minutes to spare on a connection he expected to make at the end of that trip. Two hours after he started an accident occurred which not only held up the train for two hours, but diminished its speed for the rest of the trip, causing the man to miss his connection by just 3 minutes. If the accident had occurred 6 miles farther on, the man would have been barely in time for his connection. How fast did the train go before and after the accident?

Literal Solutions

It is possible to make a general solution of a typical problem and then to use the result as a formula for the solution of other problems of the same type. There are many business and industrial situations where this procedure is followed. The formulas to be derived below are not important in themselves, but the method of deriving them is important. In solving the problems use a tabular arrangement of data whenever it is helpful.

Fundamental principle. *When you find it difficult to think in general terms, make a simple numerical illustration.*

1. Mr. A takes long walks for the good of his health. On an afternoon when he has $5\frac{1}{2}$ hours at his disposal, how many miles can he ride into the country at 30 miles an hour if he is to walk back at the rate of 3 miles an hour?

2. A man has b hours at his disposal. How far can he ride out in an automobile at c miles an hour if he is to get back in time, walking at the rate of d miles an hour? Use the result as a formula for the solution of problems 1, 3, and 4.

3. A man has five hours at his disposal. If he is to ride at 10 miles an hour and walk back at four miles an hour, how far can he go?

4. Mr. B has $6\frac{1}{4}$ hours between trains. If he hires a bicycle and rides out into the country at 9 miles an hour, then leaves the bicycle and walks back at $3\frac{1}{2}$ miles an hour, how far into the country can he go and still catch his train?

5. A can do a piece of work in 4 days and B can do it in 5 days. How long will it take them to do it together?

6. If A can do a piece of work in a days and B can do it in b days, how long will it take them to do it if they work together? Use the result as a formula for the solution of problems 5 and 7.

7. L and Z work together on a certain job. Alone L could do it in $12\frac{1}{2}$ days or Z could do it in 15 days. How long does it take them together?

8. If a bushel of corn is worth 56 cents and a bushel of wheat \$1.40, how many bushels of each must be mixed to make 12 bushels worth \$1.05 a bushel?

9. If a bushel of corn is worth c cents and a bushel of wheat w cents, how many bushels of each must be mixed to make b bushels worth p cents a bushel? Use the result to solve problems 8 and 10.

10. How many pounds of 30-cent coffee must be mixed with 45-cent coffee to secure 120 pounds that can be sold for 42 cents a pound?

11. A part of \$7000 is invested at 5% and the remainder at 4%. The total income is \$320 for a year. How much is invested at each rate?

12. A part of \$$S$ is invested at a% and the remainder at b%. The total income for a year is i dollars. How much is invested at each rate? Use the result as a formula for the solution of problems 11 and 13.

13. The annual income from \$6400 invested part at 5% and part at $6\frac{1}{2}$% is \$365. How is the money divided?

14. The sum of two numbers is s and their difference is d. What are the numbers?

15. The sum of two numbers is s and the quotient q; find the numbers.

16. Find a number which when added to both terms of the fraction $\frac{a}{b}$ gives the value c.

17. What is the nth term of the sequence $a, a + d, a + 2d \ldots$? of the sequence $a, ar, ar^2 \ldots$?

***18.** If A gives B c dollars, A will have $\frac{1}{2}$ as much as B. If B gives A c dollars, A will have 3 times as much as B. How much has each?

19. In A years Mary will be d times as old as John; B years ago Mary was c times as old as John. How old are they now?

20. Use the Pythagorean fact to derive a formula for the area of an equilateral triangle which is s inches on a side.

***21.** A baseball diamond is m feet square. How far must the catcher be able to throw to reach second base?

***22.** A and B can do a piece of work in t days; B and C can do it in s days; and A and C can do it in w days. In how many days could each do it when working alone?

***23.** A man can walk n miles an hour uphill and p miles an hour downhill. If he walks m miles up and down hill in t hours, how much of his journey is uphill?

***24.** A man can row m miles downstream in h_1 hours and m miles upstream in h_2 hours. Find the rate at which he rows in still water, and find the rate of the stream.

***25.** Use the formula of example 24 to find the rate at which a man can row in still water if it takes him 3 hours to row 6 miles upstream and only 36 minutes to row the same distance back. Find also the rate of the stream.

***26.** A two-car garage cost d_1 dollars to build and rents for d_2 dollars a year. If the owner pays taxes amounting to p dollars and spends r dollars for repairs, what per cent does the garage pay on his investment?

***27.** A certain pamphlet costs c cents per copy with a reduction of p_1 per cent when sold by the hundred up to 1000 copies and p_2 per cent when sold by the hundred over 1000 copies. How much will 500 copies cost? How much will 1500 copies cost? How much will t thousand copies cost?

***28.** Set a selling price on an article costing d dollars if 20% of the selling price is to go for overhead and 20% for profit. Under such conditions at what price would a piano sell that cost \$200 to manufacture?

***29.** A tank has three pipes; the first pipe fills it in half the time that the second takes, and the second takes only $\frac{2}{3}$ of the time that the third takes. Together they require h hours to fill the tank. How long would it take each pipe to fill the tank? If $h = 6$, how long would it take each pipe to fill the tank?

Try Exercise 44, C, page 425.

Reviews and Tests

Test A. Test of Understanding

1. In Chapter I what three fundamental notions of algebra are discussed?

2. What meanings are given to $x^{\frac{1}{3}}$, y^0, and z^{-2}? Why?

3. Express in exponential symbols, $\log_{10}100 = 2$.

4. In Chapter IV there are suggested what steps of an algebraic study?

5. On what condition may the formula $s = \dfrac{a}{1 - r}$ be used for the sum of a geometric progression?

6. In this chapter what six steps are suggested for the solution of verbal problems?

7. Explain each of the following causes of failure in problem solution:

1. Using too many of the given relations in representing the numbers and thereby not saving any relation for the equation.
2. Attempting to write the equation before the numbers and their relations are clearly in mind.
3. Confusion of units.

Test B. Problems of Distance, Rate, and Time

1. An automobile makes a trip of 150 miles at a uniform average speed. Returning over the same route it travels at a speed $2\frac{1}{2}$ miles an hour faster and covers the distance in 40 minutes less time. Find the speed at which the car started on its journey.

2. A boy makes a journey of 30 miles. He travels the first half at a uniform rate, and then proceeds for five miles at double this rate. He is then forced to finish at one mile an hour less than the rate at which he started out. Find that rate if the second half of the trip takes one hour and forty minutes longer than the first half.

3. A man walked 12 miles at a certain rate of speed and then 6 miles farther at a rate one half mile an hour faster. Had he walked the whole distance at the faster rate, his time would have been 20 minutes less. Find his original rate.

4. A train traveling from A to B, a distance of one hundred miles, reaches a point 20 miles from B when it meets with an accident and is forced to continue at half speed. As a result it reaches B one hour late. Find its usual rate.

5. A man who can row twice as fast downstream as upstream rows 12 miles down and 12 miles back in $4\frac{1}{2}$ hours. Find the rate in still water and also the rate of the stream.

Test C

1. How many pounds of nuts worth 25 cents a pound must be mixed with 20 pounds of nuts worth 20 cents a pound to make a mixture worth 22 cents a pound?

2. A and B together can do a piece of work in 10 days. Both work 7 days; then A stops and B finishes the work in 5 days more. How long would it take each to do the entire job working alone?

3. A and B working together can do a certain job in eight days. If A works 3 days and B works 6 days, half the job will be done. How long would it take for each to do the work alone? Check your answers.

4. A certain sum of money at 6% simple interest is lent for a certain time, and the interest exceeds the loan by $160. The same sum at 4% for half the time would exceed the interest on it by $480. What is the sum?

***5.** A and B are to work two days, making 24 boxes a day. On the first day A arrives first on the job and has 4 boxes finished when B appears. Working together they finish the other 20 boxes 3 hours after A started. On the second day A arrives first again and works for 2 hours. Then B appears and working alone finishes the remaining boxes in 2 hours and 40 minutes. Find the number of boxes each can make per hour.

Test D

1. A certain number of two digits is divided by the sum of those digits and the quotient is 7, the remainder 6; three times the units digit is six less than the product of the digits. Find the number.

2. A grocer wishes to mix sugar worth 9 cents a pound with sugar worth 12 cents a pound so as to procure 60 pounds of a mixture that he can sell for 11 cents a pound. What quantity of each grade of sugar must he use?

3. Twenty pounds of salt water 8% of which by weight is pure salt is to be reduced to a 5% solution. How much fresh water must be added?

4. A party of boys bought a canoe for \$70, dividing the expenses equally. Had there been two fewer boys, the expense per boy would have been \$1.75 greater. How many boys were there?

5. A dealer buying a certain number of grapefruit for \$1.04 had to throw away four bad ones. The others he sold at 6 cents apiece more than he paid for them and made a total profit of 22 cents. How many did he buy?

Test E. Progressions

1. State and derive three formulas for A.P.

2. State and derive three formulas for G.P.

3. Define the *sum of an infinite series.* Do the sums of all infinite series have limits? (Illustrate your answer.)

4. Are the following series arithmetic, geometric, or neither?

$$\tfrac{1}{4} + 1 + 4 + \dots \qquad\qquad \tfrac{1}{8} + \tfrac{1}{4} + \tfrac{3}{8} + \tfrac{1}{2} \dots$$

5. Find the sum to 6 terms of $6\tfrac{1}{4} + 2\tfrac{1}{2} + 1 + \dots$ (Either indicate the solution or compute it to four-figure accuracy.)

*6. Find the amount of $200 at 4% compounded semi-annually for three years.

*7. A debt of $24,000 is to be paid off in monthly installments of $1000 and 6% interest. The first payment is to be made at the end of the first month. What is the total amount to be paid?

8. What distance will a rubber ball travel before coming to rest if it is dropped from a height of 18 ft. and on each rebound rises $\frac{1}{3}$ of its previous height? If it is thrown to a height of 18 ft.?

Test F. General Review Test

1. Unite as indicated: $2(a+b)+3(a+b)$

$(a-x)+6(a-x)$ $\qquad$ $3(2a-b)-(2a-b)$

$a(x-5)+b(x-5)$ $\qquad$ $a(x-6)-(x-6)$

2. Factor: $\pi r^2+\pi rl$ $\quad 6a^2+3a-18$ $\quad 4x^2-20xy+25y^2$

$6x^2-xy-y^2$ $\quad x^2-1.8x+.81$ $\quad 1-5m^2+4m^4$

3. Find the sum of the first eight terms of the progression: $20+16\frac{2}{3}+13\frac{1}{3}+\dots$

4. How many terms of the progression $39+36+33+\dots$ must be taken to give a sum of 264?

5. Using the formula $A=p(1+r)^n$ find the amount if a principal of $1650 is invested at 5% compounded annually for 15 years. Use logarithms and work to the greatest degree of accuracy consistent with the tables.

6. With the help of a table of logarithms, find the value of

$$\frac{1.69^5 \times \sqrt[3]{31.0}}{0.684}$$

7. How tall is a tree if it casts on horizontal ground a shadow 72.8 ft. long when the angle of elevation of the sun is $40°50'$?

8. Find the sum of the first 20 terms of the progression 3, 9, 27 . . . as accurately as you can by logarithms.

***9.** A company puts aside $2500 at the beginning of each year in a "sinking fund." If the money earns 5% interest compounded annually, what will the fund amount to at the end of 12 years?

***10.** Two pipes together can fill a tank in h hours. One pipe running alone requires a more hours than the other to fill the tank. How many hours will it take each pipe alone to fill the tank? Make a problem to be solved by the formula which is your answer.

Test G. Review Test

1. Find the value of $\dfrac{\sqrt[3]{3897} \times \sqrt{0.1496}}{0.3265^2 \cos 81.13°}$

2. Find a binomial factor of $x^2 - 3x - 40$ which is a factor of $x^4 + 2x^3 - 43x^2 - 80x + 300$ and check your answer.

3. The expression $\dfrac{y}{5} + \dfrac{10}{y}$ is equal to $9 \div 2\frac{1}{4} + \frac{3}{6}$ for what values of y?

4. How many gallons of water must be added to 40 gallons of milk containing 4% butter fat so that the resulting mixture will contain 3% butter fat?

***5.** A garden measures 80 by 160 feet. It is desired to enlarge it by adding a border of uniform width on all sides so that the area of the border shall be 12% of the area of the garden. How wide shall the border be made? (Work to two-figure accuracy.)

6. Flying with the wind a bird was able to fly 130 miles an hour; but flying against a wind of only $\frac{1}{3}$ the velocity it could make only 50 miles an hour. Find the rates of the wind and of the bird.

7. If s represents the surface of a sphere and r the radius, $r = \sqrt{\frac{s}{4\pi}}$. Solve this formula for s and for π.

8. Write without negative exponents $\frac{sy^{-2}}{s^{-1} + y^{-3}}$.

9. Solve for a the formula $s = \frac{ab}{d - ae}$. To find whether s increases or decreases when a increases and b, d, and e remain unchanged, proceed as follows: (1) Write $s = \frac{(\quad)4}{5 - (\quad)2}$, giving arbitrary values to b, d, and e, and leaving blanks for a. (2) Supply the missing numbers in the table below:

$$a = \tfrac{1}{2},\quad 1,\quad 2,\quad 5$$
$$s = \tfrac{1}{2}$$

Test H. Equations

1. Solve and check: $\frac{2y+1}{y+3} = \frac{9 - 3y - y^2}{y^2 + y - 6} - \frac{3y - 7}{2 - y}$

2. $\frac{y - a}{2y - a} - \frac{3y - c}{6y - c} = 0$

3. $\frac{3x+1}{4} - \frac{x + 22}{12} = 3$

4. $\frac{x}{b} + \frac{y}{c} = \frac{b + c}{bc}$ $\qquad \frac{b^2 - c^2}{bc} = x - y$

5. $(2a + 1)^2 = (a - 1)^2 + 45$

6. $\frac{y^2}{12} = \frac{y}{3} + \frac{3}{4}$ (Answer to the nearest hundredth.)

7. In the equations (1) $x + 5y = 4$ and (2) $x^2 + y = 38$, solve (1) for x and substitute the resulting value in (2), and find numerical values for x.

In preparing for the next chapter, review **Exercises 11, 13, 14**, pages 358, 360.

CHAPTER VI

SPECIAL PRODUCTS AND FACTORS

Introduction

In the present chapter and the next one, the emphasis changes from the *uses* of algebra to the manipulation of algebraic expressions. Much of the material is introduced for the purpose of review and in order to give you a more complete and systematic idea of the subjects discussed. You should study with this in mind. Re-read paragraphs 2, 4, and 7 of *How to Study Algebra*, page xx.

Preliminary Test

Factor:

1. $am - an$ **2.** $3a^2b - 6ab^2 + 9ab$

3. $a(m - n) + b(m - n)$ **4.** $a(x - y) - (x - y)$

5. $x^2 + 5x + 4$ **6.** $x^2 - 3x - 4$

7. $2x^4 - 3x^2 - 5$ **8.** $3x^2 - 21x + 36$

9. $a^2 - b^2$ **10.** $150m^2 - 54n^2$

11. $(a + b)^2 - c^2$ **12.** $(a + b)^2 - (x - y)^2$

13. Factor such of the following expressions as are in the form $a^2 - b^2$:

$9 - 25c^2$ $100 - 4$ $36 - 1$ $c^4 - d^4$ $c^6 + d^6$ $(a - b)^2 - 4$
$(c + d)^2 - c^2$ $(x - 3)^2 - (x + 5)^2$ $(x - 5)^2 + (x - 4)^2$

14. How many terms are there in the square of a binomial? Among the following expressions select those which are perfect trinomial squares and modify the others so that they become such:

$a^2 + 2ab + b^2$	$a^2 + 4ab^2 + 4b^4$	$x^2 + 6x + 10$	$x^2 + 9$
$4x^2 - 4x + 16$	$x^4 + 25$	$x^2 + xy + y^2$	$x^4 + x^2 + 1$
$2x^2 - 8xy + 8y^2$	$4x + 2x^{\frac{1}{2}} + 1$	$(a+b)^2 + 3a + 3b + 1$	

15. What are some of the uses of factoring in arithmetic? In answering, consider the following operations:

(*a*) $\frac{7}{22} - \frac{5}{33}$ (*b*) $\frac{12}{5} \times \frac{5}{18}$

(*c*) $\sqrt{324} = \sqrt{2 \times 2 \times 3 \times 3 \times 3 \times 3} = 2 \times 3 \times 3 = 18$

16. What are some of the uses of factoring in algebra? In answering, consider the following:

(*a*) Express in lowest terms $\dfrac{ax - ay}{x^2 - y^2}$.

(*b*) Solve by factoring $x^2 - 5x + 6 = 0$.

(*c*) Rewrite in more compact form $A = \frac{1}{2}ah + \frac{1}{2}bh$.

(*d*) Rewrite in more convenient form for logarithmic computation

$$a = \sqrt{c^2 - b^2}, \qquad \pi r^2 - \pi r_1^2 \text{ (area of a ring).}$$

*(*e*) In the triangle *abc*, the area,

$$K = \frac{a}{2}\sqrt{b^2 - \left(\frac{a^2 + b^2 - c^2}{2a}\right)^2}$$

If $s = \frac{1}{2}(a + b + c)$ = the semi-perimeter, explain the following simplification of the formula:

$$S = \frac{a}{2}\sqrt{\left(b + \frac{a^2 + b^2 - c^2}{2a}\right)\left(b - \frac{a^2 + b^2 - c^2}{2a}\right)}$$

$$= \frac{a}{2}\sqrt{\frac{2ab + a^2 + b^2 - c^2}{2a} \cdot \frac{2ab - a^2 - b^2 + c^2}{2a}}$$

$$= \frac{a}{2}\sqrt{\frac{(a+b)^2 - ?^2}{2a} \cdot \frac{c^2 - (?)^2}{2a}}$$

$$\equiv \frac{a}{2}\sqrt{\frac{(a+b+c)(a+?-?)(c+a-b)(c-?\quad ?)}{4\,a?}}$$

$$\equiv \sqrt{s(s-a)(s-b)(s-c)}.$$

What we mean by a factor. Is $(\sqrt{x}-1)(\sqrt{x}+1)\equiv x-1$? Are we then to call the first two binomials the factors of the last binomial? The answer is found in the following statement. *To factor a rational integral expression means ordinarily to find its rational, integral prime factors.* Explain. An integral expression is said to be *prime* if it is the product of no rational, integral expressions except itself and 1. In the sense in which we are using the term, have we factored 12 when we write $12\equiv 1\times 12$ or $12\equiv 3\times 4$? Have we factored x when we write $x\equiv\sqrt{x}\times\sqrt{x}$? Have we factored x^2-25 when we write $(x-5)(x+5)$? The expression x^2-3 is prime for the purposes of this course. We do not write $(x-\sqrt{3})(x+\sqrt{3})$ as its factors.

How to study factoring. How do you know that 2 and 7 are factors of 14 and that 13 is prime? Would you know these facts if you did not know the multiplication tables of arithmetic? In the same way, the ability to factor algebraic expressions is dependent upon the recognition of certain type *products*. The steps in a study of factoring are these:

(1) Build up type products by multiplication.

(2) Learn to recognize the simple type products.

(3) Generalize the forms of the type products so that you can recognize them even when they are somewhat disguised.

With the preceding steps in mind, review the factoring of examples 1–12 above. What type products are there illustrated? By what complications are the type products disguised in 7, 8, 10, 11, 12?

Probably the most important lessons to be learned from the study of this chapter are lessons in *classification* and in *generalization*.

Type Products and their Recognition

Factors		*Product*	*Characteristics for recognition*
		I	
A monomial and a polynomial.			Any polynomial in which a monomial factor is common to all the terms.
$a(b+c-d)$	$\equiv$	$ab+ac-ad$	
$3(2x^2-5x-2)$	$\equiv$	$6x^2-15x-6$	
$5a(a+ab-1)$	$\equiv$	$5a^2+5a^2b-5a$	
		II	
Two binomials with dissimilar terms.			Four terms which contain no common monomial factor but which can be grouped in pairs so that the two terms of each pair have a common monomial factor.
$(a+b)(x-y)$	$\equiv$	$ax-ay+bx-by$	
$(3c-5d)(2a+3b)$	$\equiv$	$6ac+9bc-10ad-15bd$	
		III	
A. Two binomials with similar but not identical terms.			Quadratic trinomial in the form $ax^2 \pm bx \pm c$ or $x^2 \pm bx \pm c$, in which b (with its sign) is the sum of two factors of which the product is ac.
$(2x-3)(4x-5)$	$\equiv$	$8x^2-22x+15$	
$(r+s)(2r-5s)$	$\equiv$	$2r^2-3rs-5s^2$	
B. The same as A except that two terms are identical.			
$(x-7)(x-2)$	$\equiv$	$x^2-9x+14$	
		IV	
Two binomials with both pairs of terms identical and connected by the same sign.			Three terms. Perfect trinomial square. Two terms are squares and the other is twice the product of their square roots.
$(a+b)(a+b)$	$\equiv$	$a^2+2ab+b^2$	
$(x+5)^2$	$\equiv$	$x^2+10x+25$	
$(x-7)^2$	$\equiv$	$x^2-14x+49$	
		V	
The same as IV except that the terms are connected by different signs.			Two terms, both squares, with different signs. The difference of two squares.
$(a+b)(a-b)$	$\equiv$	a^2-b^2	
$(x+8)(x-8)$	$\equiv$	x^2-64	

Study the type products II–V until you can tell when the product of two binomials will consist of four terms, three terms, or two terms.

For type product VI see page 210. For the kinds of complications which serve to disguise these factorable types, see page 209.

Studying the Type Products

Type I. A Monomial Factor Common to All Terms

1. How is the factorable type, a monomial factor common to all terms, recognized? How many terms has it? Of what characteristics?

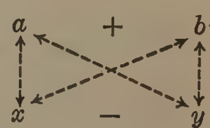
2. Factor the right-hand side in each illustrative example (page 204) of this type and check by multiplication. Notice that *the monomial factor need not be prime.*

3. Observe that this kind of factorization may be thought of as the collection of the coefficient of the monomial factor. Explain.

Type II. The General Product of Two Binomials

1. How is the factorable type, the general product of two binomials, recognized?

2. With the help of the diagram, tell which terms may be called straight products and which cross products.



3. Factor the right-hand member in each illustrative example (page 204) of this type and check by multiplication.

4. Factoring in this way is sometimes called factoring by grouping. Why?

5. Expressions of this type are sometimes said to have a common binomial factor. Study the second form of the expression below and tell why this description is appropriate.

$$ar - 2\,as + br - 2\,bs \equiv a(r - 2\,s) + b(r - 2\,s) \equiv (a + b)(r - 2\,s).$$

Notice that the final step in the factorization is in fact the collection of the coefficient of this second binomial. For practice in this sort of addition, see Exercise 1, D, page 350.

6. Factor the preceding expression by grouping the first term with the third and the second with the fourth.

7. Factor the two expressions below and explain the complications which make the factoring more difficult than that in the models.

$$axb + acx - aby - acy, \quad ar + br - cr - as - bs + cs.$$

Type III. The Quadratic Trinomial

1. How is the factorable type, the quadratic trinomial, recognized?

2. Factor the right hand side of each illustrative example (page 204) of this type and check by multiplication. (See page 32.)

3. Verify the following identity:

First form		*Second form*		*Third form*
$(3\,x - 5\,y)(4\,x - y)$	$\equiv$	$12\,x^2 - 3\,xy - 20\,xy + 5\,y^2$	$\equiv$	$12\,x^2 - 23\,xy + 5\,y^2$

This trinomial can be factored by the method illustrated on page 32. Another method is to rewrite it in the second form and to factor it by grouping in pairs. In order to factor the third form, let us retrace the steps by which it was obtained. Thus, in order to obtain the $-3\,xy$ and the $-20\,xy$, multiply 12 by 5 and get 60; take the two factors of 60 of which the sum is -23; namely, -3 and -20. Now rewrite the second form and factor it.

In the same way factor $3x^2 - 10x - 25$ and $10x^2 + 19x + 6$. Factorable quadratic trinomials in the form $ax^2 + bx + c$ may then be factored by the following steps. Explain and illustrate.

(1) Find two numbers of which the product is ac and the algebraic sum b.

(2) Replace bx by two terms having for coefficients the numbers just found.

(3) Factor by grouping in twos.

4. Factor $4x^2 + 5xy - 6y^2$. $4(-6) = -24 \equiv 8(-3)$. (These two factors of 24 are selected because their sum is what?)

$$4x^2 + 8xy - 3xy - 6y^2$$

Complete the factorization and check by multiplication.

Type IV. The Perfect Trinomial Square

1. How is the factorable type, the perfect trinomial square, recognized?

2. Factor the illustrative examples (page 204) of this type and check by multiplication.

3. Factor $25z^2 - 60zb + 36b^2$.

4. Factor $100a^2b^2 - 120abc + 36c^2$ and tell what complications are involved.

Type V. The Difference of Two Squares

1. How is the factorable type, the difference of two squares, recognized?

2. In general the product of two binomials contains four terms. On what condition does it contain but three? But two?

3. Factor $4z^2 - b^2$, $9m^2 - 64b^2c^2$.

4. Factor $x^2 - \frac{4}{y^2}$. Notice that the factors will not be integral in y.

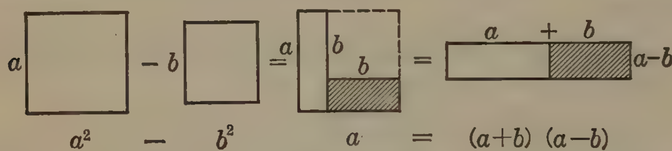
5. Factor $ax^2 - 25\,ay^2$ and comment on the complication involved.

Try Exercise 45, page 426, **and Exercise 13,** page 360.

More Complex Factorable Expressions

1. Recognition of frequently recurring algebraic forms. What are the three steps in the study of factoring as outlined in this chapter? Why is it important to learn to classify algebraic expressions by means of the forms in which they can be expressed?

2. Our fifth type product would have been represented by the early Greeks, who were the discoverers of much of our secondary school mathematics, by such *geometric* forms as the following, because early Greek algebra was expressed by means of diagrams.



Even when using the efficient modern symbolism of algebra, geometric figures are sometimes helpful as you have found, for instance, in your study of problem solution.

Certain types of algebraic expressions are also recognizable by their *algebraic* forms. Our fifth type product, for instance, may be represented in the following form even when it is complicated by the substitution of binomials for its terms:

$$\{[\quad] - (\quad)\}\{[\quad] + (\quad)\} \equiv [\quad]^2 - (\quad)^2:$$

Explain and illustrate.

Construct similar diagrams for type products IV and V. It is important to learn to classify algebraic expressions by their *algebraic* form.

Various complications in factorable expressions. The following examples illustrate some of the complications which may serve to disguise factorable expressions and make them less easy to recognize.

3. Factor by the use of the appropriate forms:

$$r^2 + s^2 - c^2 + 2\,rs$$

Analysis: (1) There is no common monomial factor.

(2) Factoring by grouping, two and two, will not work as can be determined by the signs and in other ways.

(3) The $-c^2$ suggests the difference of two squares. The $2\,rs$ suggests that the terms containing r or s may form a perfect square. The diagrammatic form for type V gives

$$[r+s]^2 - c^2 = \{[\quad] - c\}\{[\quad] + c\}$$

Complete and check by multiplication or by numerical substitution.

4. $r^2 + 2\,rs + s^2 - c^2 - 2\,cd - d^2$

5. $2\,cd - 2\,ab + a^2 - c^2 + b^2 - d^2$

6. $(a+b)^2 - 2(a+b)c + c^2$

7. $(a+b)(a-b)^2 - 2(a^2 - b^2) + a + b$ Hint: First divide by $(a+b)$.

8. $x^4 - 10\,x^2 + 9$. Factor as a quadratic trinomial in x^2.

9. $x^4 - 7\,x^2 + 9$

Analysis: (1) There is no common monomial factor.

(2) The quadratic trinomial method fails because no two factors of 9 have -7 for their sum. (3) Reduce to the difference of two squares thus:

$x^4 - 7\,x^2 + 9 \equiv x^4 - 6\,x^2 + 9 - x^2 \equiv (x^2 - 3)^2 - x^2 = ?$ Explain.

10. $4\,x^4 - 44\,x^2y^2 + 49\,y^4$. Suggestions: First add and then subtract $16\,x^2y^2$

11. $25a^4 + 26a^2b^2 + 9b^4$ **12.** $25a^4 - 19a^2 + 9$

***13.** $x^{2a} + 5x^a + 4$ ***14.** $a^by^2 + 2a^by + a^b$

***15.** $y^2 + 5(5 - 2y)$

16. Among the complications which may serve to disguise factorable expressions are the following. Point out an illustration of each: Literal exponents. The combination in one expression of two or more factorable types. The replacement of a letter by a less simple expression. Which of these complications suggests the following caution? ***Examine each factor in order to discover whether it is prime.***

Other Type Products

Type Product VI. The Sum or the Difference of Two Cubes

A. $(a+b)(a^2 - ab + b^2) = a^3 + b^3$

B. $(a-b)(a^2 + ab + b^2) = a^3 - b^3$

***1.** Test A and B by both multiplication and division and commit them to memory. Give particular attention to the signs. Translate each identity into words. Explain the significance of the names of this pair of types and tell how they may be recognized. Discuss and illustrate complications which may occur in this factorable type.

Factor and check by multiplication:

***2.** $a^3 - 8b^3 = (a - 2b)(a^2 + ? + ?)$

***3.** $x^6 + 27y^9 = (x^2)^3 + (3y^3)^3 = (x^2 + 3y^3)(x^4 - \quad)$

***4.** $x^3 + y^3$ ***5.** $x^3 - y^6$ ***6.** $27^3 - 8y^3$

***7.** $x^6 - y^6 =$ (by type V) $(x^3 + y^3)(x^3 - y^3) \equiv$ (by type VI) $(x+y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2)$

But also $x^6 - y^6 = (x^2 - y^2)(x^4 + x^2y^2 + y^4) \equiv (x-y)(x+y)(x^2 + y^2 - xy)(x^2 + y^2 + xy)$

*8. $x^6 - 64$ *9. $x^3y^6 - 64$

The zero-product principle.

10. $(x + 2)(x - 3)(x + 4) = x^3 + 3x^2 - 10x - 24$. Verify.

Products of this type may be factored by means of the principle stated in Example 19, page 17. It is evident that if the left member of the preceding identity is set equal to zero, the resulting equation will be satisfied if $x + 2 = 0$, or if $x - 3 = 0$, or if $x + 4 = 0$; that is, if $x = -2$, or 3, or -4.

To factor this product, substitute for x the factors of 24 in turn until a number is found which reduces the expression to zero. 3 is such a factor of 24, hence $x - 3$ is one of the factors of the expression. Divide by this factor, and factor the quotient.

*11. Factor $x^3 - 2x^2 - 5x + 6$.

*12. Factor $x^3 + 2x^2 + 3x - 6$.

*13. Is $x^5 - 1$ divisible by $x - 1$? by $x + 1$?

*14. Is $x^5 - 32$ divisible by $x - 1$? by $x + 1$? by $x - 2$? by $x + 2$?

*15. Find a binomial factor of $x^5 + 1$. Of $x^5 + 32$.

*16. Is $x - 1$ a factor of $x^4 - 1$? Is $x + 1$?

*17. Is $x - 2$ a factor of $x^4 - 16$? Is $x + 2$?

*18. By a method suggested by the preceding examples, discover a binomial factor of $x^5 - y^5$. Of $x^5 + y^5$. Of $x^4 - y^4$.

How to Develop Skill at Factoring Algebraic Expressions †

I. Fix in mind the five or six type products and the kinds of complications which commonly serve to disguise them.

† Skill in factoring is now given much less prominence than formerly in courses in algebra.

II. Remove any monomial factor which is common to all terms.

III. Classify according to the number of terms:

1. Binomials, (*a*) difference of two squares,
   (*b*) sum or difference of two cubes.
2. Trinomials, (*a*) perfect squares,
   (*b*) other quadratic trinomials.
3. Polynomials of four terms, (*a*) group, two and two, and take a common factor from each pair,
   (*b*) group, three and one, and factor as the difference of two squares.
4. Polynomials of six terms, (*a*) group, three and three, and factor as the difference of two squares,
   (*b*) group, two, two, and two, or three and three, and take monomial factors common to the terms of each group.

IV. Caution. Examine each factor in order to see if it is prime. Authors of examinations and of textbooks often attempt to test your alertness by so preparing factorable expressions that the first factors obtained are not prime.

Try Exercise 46, page 429.

Some Uses for Factoring

Review Exercises 15 and 16, page 202.

1. The transformation of formulas. The total area of the surface of a circular cylinder may be found by use of the formula $t = 2\pi r^2 + 2\pi rh$. Express this formula in a compact and convenient form. Find the surface of a cylinder 10 feet high with a base 4 feet in radius. Use $\frac{22}{7}$ for π.

*2. Use the formula of the preceding example in finding the total surface of a cylinder in which $r = 1.4'$ and $h = 2.1'$. Use 3.1 for π and work to three-figure accuracy, rounding off your answer to two-figure accuracy.

3. The amount to which a certain principal grows at a certain rate of simple interest in a given time may be found by use of the formula $A = p + prt$. Express the formula in more compact form. Find how much \$150 will amount to in 12 years at 5% simple interest.

*4. Use the formula of Example 3 in finding the value of \$530 after 7 years at $4\frac{1}{2}\%$ simple interest.

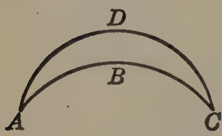
5. The area of a flat circular ring may be found by use of the formula $A = \pi r^2 - \pi R^2$. Express this formula in factored form. Find the area of a ring in which r, the radius of the outer circle, is $22''$ and R, the radius of the inner circle, $18''$. Use $\frac{22}{7}$ for π.

*6. Use the formula of Example 5 in finding the area of a circular walk $14'$ wide around a circular pool $50'$ in diameter. Work to three-figure accuracy and round off your answer to two-figure accuracy.

7. Write compact formulas for: (*a*) The volume of a hollow cylinder; that is, a cylinder through which a cylindrical hole has been bored lengthwise. (The volume of a cylinder, $V = \pi r^2 h$.) (*b*) The area which remains if a square of side s is cut from each side of a square whose side is S. (*c*) The area of the four side walls of a room l feet long, w feet wide, and h feet high. (*d*) The area of a trapezoid, $\frac{1}{2}hb + \frac{1}{2}hb'$. (*e*) The volume of a spherical shell, $\frac{4}{3}\pi r^3 - \frac{4}{3}\pi R^3$, where r is the radius of the outer sphere and R the radius of the inner.

*8. Find the area of a trapezoid of which the shorter base is $12.5''$, the longer base $18.3''$, and the altitude $9.7''$, using formula (*d*) of Example 7.

*9. The area of the crescent-shaped figure $ABCD$ (bounded by a semi-circle and an arc of twice the radius of the semi-circle) can be proved to be equal to $2\pi r^2 - \left(\frac{\pi r^2}{6} + \frac{r^2}{4}\sqrt{3}\right)$ where r is the radius of the semi-circle. Find the area of such a crescent when $r = 5$ inches, $\pi = 3.14$.



The reduction of fractions. Reduce each of the following fractions to lowest terms by dividing numerator and denominator by common factors until no common factor remains.

10. $\dfrac{a^2 - b^2}{a - b}$

11. $\dfrac{m^2 - m^3}{m^2 - 1}$

12. $\dfrac{1 - 2\,m}{8\,m^2 - 2}$

13. $\dfrac{x^2 - 1}{x^2 + 3\,x + 2}$

14. $\dfrac{x^2 - 9\,y^2}{2\,ax + 6\,ay}$

15. $\dfrac{m^4 - n^4}{m^4 - 5\,m^2n^2 - 6\,n^4}$

***16.** $\dfrac{3\,a^3 + 3\,a^2 - a - 1}{3\,a^2 + 9\,a + 6}$

17. $\dfrac{16 - (a + b)^2}{b^2 - (a + 4)^2}$

***18.** $\dfrac{a^2 - 2\,ab + b^2 - m^2}{3\,a^2 - 3\,ab + 3\,am}$

19. $\dfrac{4\,a^2 - 9\,b^2}{2\,a^2 - 5\,ab + 3\,b^2}$

***20.** $\dfrac{4\,a^2 - 8\,ab + 4\,b^2}{4\,a^3 - 4\,a^2b - 4\,ab^2 + 4\,b^3}$

***21.** $\dfrac{x^2 - 2\,x + 1}{x^3 - 1}$

22. A fraction may be reduced to lowest terms in one step by dividing its numerator and denominator by their *highest common factor* (abbreviated H.C.F.). Explain the process, and explain the significance of each italicized word.

23. Finding a lowest common multiple. In order to add $\frac{2}{3}$ and $\frac{3}{4}$ we express both fractions as twelfths. Why? Give other common denominators which might be used instead of 12.

Twelve is called the *least common multiple* of 3 and 4 (abbreviated L.C.M.). Give several multiples of 3, of 4. Give several common multiples of 3 and 4. Show that there is no number which is the greatest common multiple of 3 and 4.

Add:

24. $\frac{2}{5}+\frac{6}{7}$ **25.** $\frac{1}{a}+\frac{2}{b}$ **26.** $\frac{a}{b^2}+\frac{c}{ab^3}$

27. Find the L.C.M. of a^2-ab, $3a^2+3ab$, and a^2-b^2

Plan of work:

$a^2-ab = a(a-b)$

$3a^2+3ab = 3a(a+b)$

$a^2-b^2 = (\quad)(\quad)$

Select for the L.C.M. enough of these factors so that their product will be divisible by each of the given expressions and yet contain no extra factors. State a rule for finding the least common multiple of two or more expressions.

Find the L.C.M. of the expressions below. Give results in factored form.

28. $a^2-2ab+b^2,\quad a^2+ab-2b^2$

29. $a^2-4b^2,\quad a^2+ab-6b^2,\quad a^2+5ab+6b^2$

30. $9a^2-9a,\quad 6a-6,\quad 3a+3$

***31.** $m^2+7mp-18p^2,\quad m^2-4p^2,\quad m^2+9mp$

***32.** $(ab-b^2)^3,\quad ab^2-b^3$

***33.** $x^3-8,\quad x^3-6x^2+12x-8,\quad 5x^2+20x+20$

Solving equations by factoring. Arrange in the form $ax^2 \pm bx \pm c = 0$, factor, solve, and check:

34. $x^2-13x+40=0$ **35.** $x^2-16=0$

36. $2x^2-7x-15=0$ **37.** $5x^2-3=-14x$

38. $x^2-4x=0$ **39.** $x^2=4x$

40. $20y^2-16y=4$ **41.** $3x^2-15=4x$

***42.** $x^3-x^2-x+1=0$ ***43.** $x^2-1+2(x-1)=0$

44. $x^{\frac{4}{3}} - 5\,x^{\frac{2}{3}} + 4 = 0,\quad (x^{\frac{2}{3}} - 1)(x^{\frac{2}{3}} - 4) = 0$

$x^{\frac{2}{3}} = 1 \quad x^{\frac{2}{3}} = 4$ By the zero factor principle.

$x = 1^{\frac{3}{2}} \quad x = 4^{\frac{3}{2}}$ Raising each member to the $\frac{3}{2}$ power.

Complete and check:

45. $x^{\frac{1}{4}} + x^{\frac{1}{2}} = 6$

46. $25\,x^{\frac{2}{3}} + 20\,x^{\frac{1}{3}} - 21 = 0$

47. $x^{\frac{4}{3}} + 7\,x^{\frac{2}{3}} = 8$

***48.** $x^{-\frac{4}{3}} - 5\,x^{-\frac{2}{3}} + 4 = 0$

***49.** $2\sqrt[3]{x^{-2}} - 3\sqrt[3]{x^{-1}} = 2$

***50.** $2\sqrt{x^3} - 3\,x^{\frac{3}{4}} = 35$

The Factor Theorem

If a (rational and integral) polynomial in x becomes zero when a is substituted for x, then $x - a$ is a factor of the polynomial.

***1.** $(x - 2)(x + 3)(x - 5) \equiv x^3 - 4\,x^2 - 11\,x + 30$. The three numbers which when substituted for x make this expression zero are 2, -3, and 5 as is evident on inspection of the left-hand form and as may be found by experiment with the right-hand form. Notice that these numbers are all factors of 30. Why? They are the only numbers which make the expression zero because they are the only numbers which make any of the factors zero. Explain. Cover up the left-hand form and factor the right.

Plan of solution: Substitute for x, factors of 30 in turn, for example $+1, -1, +2, -2$, etc., until the expression becomes zero. Set down the corresponding factor, $x - 2$, in this case. (See the theorem above.) Divide the polynomial by this factor and if possible factor the quotient by the factor theorem or by any other method.

Factor if possible:

***2.** $x^3 - x^2 - 9\,x + 9$

***3.** $x^3 - 2\,x^2 - 9$

***4.** $3\,x^3 + 15\,x^2 - 12\,x - 60$

***5.** $y^3 - 9\,y^2 + 26\,y - 24$

*6. $x^3 + 3x^2 - 4x - 12$ *7. $y^3 - 5y^2 - 2y + 24$

*8. $z^3 - 3z^2 - 3z - 4$ *9. $x^4 + 5x^3 + 5x^2 - 5x - 6$

*10. $x^3 + 4bx^2 - 3b^3$ (Substitute $-b$ for x)

*11. $x^3 - 4bx^2 + 5b^2x - 2b^3$ *12. $x^3 - 4ax^2 + ax + 6a^3$

*13. $x^3 - 8$ *14. $x^3 - 27$ *15. $x^5 - 1$ *16. $x^4 + 16$

*17. $x^4 - 16$ *18. $x^5 - 32$ *19. $x^3 - y^3$ (Put $-y$ and y for x)

*20. $x^4 - y^4$ *21. $x^4 + y^4$ *22. $x^5 - y^4$ *23. $x^5 + y^5$

Tests

Test A

Answer Questions 1–6, Exercise 46, A, page 429.

Using a systematic plan of procedure, factor all the following expressions which are not prime.

1. $6x^2 - 5x - 6$
2. $x^2 + 9y^2 - 25z^2 - 6xy$
3. $4x^2 - 0.25$
4. $ax^2 - 2ax - 24a$
5. $3a(x - 2y) - 4b(2y - x)$
6. $a^2 - b^2 + 2b - 1$

*7. $(x - 2y)x^3 - (2y - x)y^3$

8. $8x^2 + 2x - 15$
9. $x^2 + 2xy + y^2 - p(x + y)$
10. $x^2 - 10x - 24$
11. $b^2 - ab + bc - ac$

*12. $3a^2 - 24a^5$

13. $3x^2 - 15x - 150$
14. $2mn - m^2 - n^2$
15. $x^2 - a^2 + y^2 - b^2 - 2xy + 2ab$
16. $a^2 - 25b^2c^4$
17. $25m^2n^2p^2 - 9$
18. $81m^5 - 16mn^4$
19. $1 - x^2 - 2xy - y^2$

*20. $r^4 + rs^3$

21. $5r^3 - 20r^2 - 300r$

*22. $64a^7 - a$

23. $r^3 + r^2s + 2rs^2 + 2s^3$
24. $r^4 - 25(m - 3)^2$
25. $4a^2b^2 - (a^2 + b^2 - c^2)^2$

Test B. To Test Your Understanding

Which of the following statements are true?

1. $a^2 - 2ab - b^2$ is a perfect square.

2. The factors of $a^2 - 9b^2$ are $(a - 3b)(a - 3b)$.

3. $1 - 2x + x^2$ is a perfect square.

4. $a^6 - b^6$ is the difference of two squares.

5. $x^6 - y^6$ is the difference of two cubes.

*6. $\dfrac{8a^3 - b^3}{2a - b} \equiv 4a^2 + 2ab + b^2$

7. The prime factors of $4a^2 + 8ab + 4b^2$ are $(2a + 2b)$ and $(2a + 2b)$.

8. $(a^2 - 2b)^2 \equiv a^4 - 4a^2 + 4b^2$

9. $\dfrac{4a^2 - 1}{1 + 2a} \equiv 2a - 1$ *10. $\dfrac{a^6 - b^6}{a^2 - b^2} \equiv a^3 + b^3$

11. $21^2 \equiv (20 + 1)^2 \equiv 400 + 40 + 1 = 441$

12. $19 \times 21 \equiv 400 - 1 \equiv 399$

13. $(a + b)(c - d) + (c - d) \equiv (c - d)(a + b + 1)$

14. The prime factors of $2x^2 - 4x - 6$ are $(2x + 2)$ and $(x - 3)$.

15. $b^2 - a^2 \equiv (a + b)(a - b)$

16. $\dfrac{49a^2 - 14ab + b^2}{7a - b} \equiv 7a - b$

17. $a^2 + b^2$ is a binomial.

18. $9m^2 + 15mn + 25n^2$ is a perfect square.

19. $m^{2a} + n^{2a} - s^{2a}$ contains a common monomial factor.

Test C

Reduce to lower terms:

1. $\dfrac{r^2 - 2\,rs - 15\,s^2}{r^2 - 4\,rs - 21\,s^2}$

2. $\dfrac{a^3 - b^3}{a^2 - ab}$

3. $\dfrac{2\,a^2 + a - 6}{a^2 - a - 6}$

4. $\dfrac{a^3 - 3\,a^2 + 2\,a}{2\,a^2 + 10\,a - 12}$

Find the H.C.F. and the L.C.M. of the following:

*5. $x^2 - 1, \quad x^2 - 2\,x + 1, \quad x^3 - 1, \quad x^2 + 2\,x - 3$

*6. $1 - x^2, \quad 15 - 16\,x + x^2, \quad 1 - x^3$

*7. $(ab - b^2)^3, \quad b^3 - b^2y^3, \quad a^3 - b^3, \quad b^2 - 2\,ab + a^2$

*8. $a^2 - b^2, \quad a^3 - b^3, \quad (a - b)^2$

Solve for x by factoring:

9. $2\,x^2 - x - 6 = 0$

10. $7\,x^2 - x = 6$

11. $6\,x^2 - 7\,ax = 24\,a^2$

12. $x^4 - 5\,x^2 = -4$

13. $2\,mx - ms = 3\,s - 6\,x$

14. $ax - 2\,bx = a^2 - 5\,ab + 6\,b^2$

Multiply as indicated:

15. $78^2 = (70 + 8)^2, \quad 78 \times 82 = (80 - 2)(80 + 2), \quad 63^2, \quad 45^2, \quad 29^2, \quad 57 \times 63, \quad 29 \times 31$

Test D. Review

Perform the operations indicated and express without negative exponents:

1. $\dfrac{3\,a^{-1}b^{-2}}{4\,x^{-1}y^{-4}} \times \dfrac{6\,a^2x^{-1}}{5\,b^{-1}c^2}$

*2. $\sqrt[3]{\dfrac{a^{-1}}{b^{-3}}} \div \left(\dfrac{a^{-1}\,b^{\frac{1}{2}}}{b^{-1}a}\right)^2$

3. $\dfrac{x^{\frac{1}{3}}\sqrt{y^{-\frac{3}{2}}}}{y^3 + \sqrt[3]{x^{-1}}}$

*4. $\dfrac{r^{-1}s^{-1}}{r^{-2} - s^{-2}} \times \dfrac{r^0}{(r + s)^{-1}}$

Estimate the values of the following fractions and then find the values by the use of logarithms:

5. $\dfrac{\sqrt{817.5}}{0.09876}$

6. $\dfrac{\sqrt[3]{4.118}}{0.4378}$

7. If a car starts down an incline and runs down 2 ft. the first second, 6 ft. the second second, 10 ft. the third second, and so on, how far will it run in 12 seconds?

8. A man has an appointment which he can meet if he starts at once and drives at the rate of 25 miles an hour. Being delayed an hour in starting, he drives 30 miles an hour and arrives 10 minutes late. At what rate should he have driven in order to have arrived on time?

9. In solving verbal problems, what are three fundamental habits to form?

10. What is the logarithm of 4 to the base 2?

11. In a right triangle $A = 32° 46'$ and $a = 56.32'$; find c.

Test E. A Midyear Examination

1. (*a*) $S = \dfrac{a - ar^n}{1 - r}$. Find S when $a = 3$, $r = 3$, $n = 5$.

(*b*) $V = \frac{1}{3}\pi r^2 h$. Find h when $\pi = 3.14$, $r = 2.30$, $V = 75.36$.

2. (*a*) Simplify: $8^{-\frac{2}{3}} - 2^{-2} + 1^{\frac{2}{5}} - 7^0$

(*b*) Simplify: $\left[\dfrac{\sqrt{72\,y^{2n}}}{3} \times 9^0\right](2\,y^{n+2})^{-1}$

3. A boat can travel 8 miles an hour in still water. If it can travel 20 miles downstream in the same time that it can travel 12 miles upstream, what is the rate of the stream?

4. A man has \$15,000 invested, a part at 4% and a part at $5\frac{1}{2}$%. The interest for one year on the 4% investment exceeds

the interest for one year on the $5\frac{1}{2}\%$ investment by \$30. How much is invested at each rate?

5. An architect who made plans for a house whose length was 10 ft. more than its width found that he had to reduce the size on account of expense. By decreasing each dimension 5 ft., he decreased the area of the floor-plan by 425 square feet. What were the original dimensions?

6. Factor completely: (*a*) $9\,c^4 - 34\,c^2d^2 + 25\,d^4$

(*b*) $r^2 - 225\,s^2 + 25\,t^2 - 10\,rt$

(*c*) $4\,x^4 + y^4$

7. Find by use of logarithms the value of $\sqrt{\dfrac{1.785 \times 0.08762}{4.114}}$

Test F. Review

1. Solve and check: $x^2 + \dfrac{7\,y}{3} = \dfrac{27}{2}$, $\dfrac{17\,y}{5} = \dfrac{15}{2} - x^2$

2. Solve and check: $\dfrac{3\,y - 3}{y + 7} + \frac{3}{5} = 0$

3. Solve and check: $\dfrac{b}{a} = \dfrac{y - a^2}{y - b^2}$

4. Find the roots to three-figure accuracy and check one of them: $x^2 + 4\frac{2}{3}\,x = 7$

5. Solve for a: $C = \dfrac{ae}{R + ar}$ If a increases and e, R, and r remain unchanged, does C increase or decrease? Show how you arrive at your answer.

6. Find by logarithms the reciprocal of the cos 35° 19′.

7. Factor: $a^2 + 3\,a + 3\,b + ab$, $8\,x^2 - 18\,x + 9$, $6\,x^{2n+2} - 24$

8. Factor: $(a) a^4 + 3a^2b^2 + 4b^4$, $*(b) x^{3a} - x^{3b}$, $(c) \dfrac{y^2 + z^2 - x^2}{2yz} + 1$

***9.** If $S = \dfrac{a}{2}\sqrt{b^2 - \left(\dfrac{a^2 + b^2 - c^2}{2a}\right)^2}$ and if $s = \dfrac{a + b + c}{2}$

prove that $S = \sqrt{s(s-a)(s-b)(s-c)}$.

10. Reduce to lowest terms:

$(a)\ \dfrac{a^2 - 3a - 10}{a^2 - 6a + 5}$, $\quad *(b)\ \dfrac{x^2 + 2xy + y^2}{x^3 + y^3}$

11. To 120 lb. of a 4% solution of a certain chemical, enough of a 10% solution is added to bring the solution up to 6%. How many pounds will there be in the total mixture?

12. Set a selling price for a table costing \$130 so as to allow 30% of the selling price for profit and 12% of the selling price for overhead. Answer to the nearest dollar.

Test G. Some "One" Cases

1. Write an arithmetic sequence in which $a = 1$ and $d = 1$.

2. Write a geometric sequence in which $a = 1$ and $r = 1$.

3. What is the sum to infinity of $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$?

4. Why is x^0 given the value 1?

5. $x^n \cdot x = ?$ $\quad x^n \div x = ?$ $\quad a^b \times a = ?$ $\quad a^b \times b = ?$

6. Add $a\sqrt{a+b} + (a+1)\sqrt{a+b} - \sqrt{a+b}$

7. $(-1)^3 = ?$ $(-1)^4 = ?$ $(-1)(a-b) = ?$ $\dfrac{(b-a)}{-3} \cdot \dfrac{(-1)}{(-1)} = ?$

8. Factor $a^3bc - a^2b^3c - a^2bc$

9. Write the reciprocal of x and show how a number and its reciprocal are related.

10. At 4% simple interest for one year x dollars will amount to x (?) dollars.

CHAPTER VII

FRACTIONS AND FRACTIONAL EQUATIONS†

PART I. FRACTIONS

Introduction

THE study of this chapter should increase your understanding of algebraic and arithmetic fractions, of their transformations, and of their uses.

1. What a fraction means. $\frac{4}{5}$ means ? ÷ ? $\frac{4}{5}$ also means ? of the ? equal parts of a thing or a group of things. A fraction may also be thought of as a ratio. Show that this third meaning is illustrated by π in the formula $\frac{c}{d} = \pi$.

2. Show that the bar of a fraction serves some of the purposes of a parenthesis. Consider, for example, the expression $\frac{a+b}{c} \times \frac{a-b}{c}$. The sloping bar which is often seen in printed fractions is not recommended for written work. $x/a + 2$ means $\frac{x}{a} + 2$: to express $\frac{x}{a+2}$ with the sloping bar, we write $x/(a+2)$. Without the parenthesis the effect of the sloping bar extends only to the first plus or minus sign. $a\frac{b}{c}$ means $a \times \frac{b}{c}$. Contrast this meaning with the meaning of $5\frac{2}{3}$.

3. Changing the form of a fraction without changing its value. *A fraction is a ratio: changes in the form of the fraction which do not change this ratio do not change the value of the fraction.* For each fraction below write at least two equivalent fractions. Test your work by the foregoing principle. Tell whether the value

† Pupils who can do well on the tests on pages 228, 229, 238 ff. can omit most of this chapter except perhaps proportion, pages 229 ff.

of a fraction is changed when numerator and denominator are both increased, decreased, multiplied, or divided by the same number. Illustrate the fact that multiplication and division by 0 must be excluded.

$$\tfrac{4}{5},\quad \tfrac{12}{18},\quad \frac{a}{b},\quad \frac{a^2}{ab},\quad \frac{a+b}{a},\quad \frac{x^2-y^2}{x-y},\quad \frac{b-a}{c}$$

4. Test the following identities or complete them:

$$\tfrac{2}{3} \overset{?}{\equiv} (\tfrac{2}{3})^2,\qquad \frac{(a-b)a-(a+b)}{a^2-b^2} \overset{?}{\equiv} \frac{a-(a+b)}{a+b}$$

$$\frac{4\frac{1}{2}}{5\frac{2}{3}} \overset{?}{\equiv} \frac{4\frac{1}{2}\times 6}{5\frac{2}{3}\times 6} \equiv ?\qquad \frac{a+\frac{b}{c}}{b-\frac{a}{c}}\times\frac{c}{c} \overset{?}{\equiv} ?$$

5. Insert $\equiv$ or $\not\equiv$ between each two of the following expressions, and if they are identical, tell how one may be transformed into the other:

$$\frac{ab}{a}\quad b,\qquad \frac{a-b+c}{a+b+c}\quad \frac{a-b}{a+b},\qquad \frac{a^2+b^2}{a-b}\quad a-b,\qquad \frac{a^2+b^2}{a+b}\quad a+b,$$

$$\frac{a+b}{a}\quad a,\qquad \frac{\sqrt{5}}{\sqrt{5}}\quad 1,\qquad \frac{5}{\sqrt{5}}\quad \sqrt{5},\qquad \frac{3}{2+\sqrt{5}}\quad 6-3\sqrt{5},$$

$$\frac{a(b-c)}{a+(b-c)}\quad 1,\qquad \frac{a^{-2}+b^{-3}}{a^{-3}-b^{-2}}\quad \frac{ab^3+a^3}{1-a^3b}$$

6. Multiplication of numerator and denominator of a fraction by −1.

Insert $\equiv$ or $\not\equiv$ between each pair of expressions below:

$$\frac{8}{-4}\quad \frac{8}{4},\qquad \frac{12}{-2}\quad \frac{-12}{-2},\qquad \frac{a}{-b}\quad \frac{-a}{b},\qquad \frac{(a-b)}{5}\quad \frac{(b-a)}{5},\qquad \frac{a-b}{b-a}\quad 1,$$

$$\frac{a(b-a)(c-b)}{s-r}\quad \frac{a(a-b)(b-c)}{r-s},\qquad \frac{-a+b}{x}\quad \frac{a+b}{x}$$

Replace each question mark below by *is* or *is not*. The value

of a fraction ? changed when the numerator (or the denominator, or the fraction itself) is multiplied by -1. The value of a fraction? changed when both numerator and denominator (or numerator and fraction, or denominator and fraction) are multiplied by -1. Show that the facts you have stated above are included in the principle of exercise 3. Show that the multiplication of a numerator (or denominator) by -1 means the multiplication of all its terms or of an odd number of its factors.

7. A contrast. In the two following operations, tell why the denominator 6 is written in the first case and not in the second: I. $\dfrac{3x-2}{3} \equiv \dfrac{6x-4}{6}$ II. ① $\dfrac{3x-2}{3} = \dfrac{4x-4}{2}$

② $6x - 4 =$ etc. ① $\times$ 6.

8. Simplify each fraction by multiplying its numerator and denominator by the same number:

(*a*) $\dfrac{4\frac{2}{3}}{5\frac{1}{6}}$ (*b*) $\dfrac{x}{x+\frac{2}{3}}$ (*c*) $\dfrac{b-a}{-3}$ (*d*) $\dfrac{a+\dfrac{a}{b}}{b+\dfrac{b}{c}}$ (*e*) $\dfrac{\sqrt{3}}{\sqrt{2}}$ (*f*) $\dfrac{a}{\sqrt{3}-5}$

(*g*) $\dfrac{\sqrt{3}}{\sqrt{2}+\sqrt{3}}$ (*h*) $\dfrac{2a^{-1}b^{-1}}{ac^{-2}}$ (*i*) $\dfrac{3}{a^{-1}+b^{-2}}$ (*j*) $\dfrac{x^{-1}-y^{-2}}{az^{-2}}$

Try Exercise 47, page 432.

Processes with Fractions

1. Addition and subtraction. Write as one fraction

$$\frac{x^2+y^2}{x^2-y^2} - \frac{x}{x+y} + \frac{2y}{y-x}.$$

The sum is

$$\frac{x^2+y^2-x(x-y)-2y(?)}{(x+y)(x-y)} \equiv \frac{-xy-y^2}{(?)\ (?)} \equiv \frac{-y}{?}$$

Complete and explain. Check by substituting 3 for x and 2 for y; thus:

The value of the original fraction	Of the answer
$\frac{9+4}{9-4}-\frac{3}{3+2}+\frac{4}{2-3}$	$\frac{-2}{3-2}$
$\frac{13}{5}-\frac{3}{5}+\frac{4}{-1}$	$\frac{-2}{1}$
$\frac{10}{5}-4$	-2
$2-4$	
-2	

2. Write as one fraction: $\dfrac{x+y}{x-y}-\dfrac{x^2+y^2}{y^2-x^2}+\dfrac{y-x}{x+y}.$

Try Exercise 48, page 435.

Multiplication and division.

3. $$\frac{a}{a-b}\left(\frac{a+b}{a-b}+\frac{a-b}{a+b}\right)\div\left(\frac{1}{a-b}-\frac{1}{a+b}\right)$$

$$\equiv\frac{a}{a-b}\times\frac{(a+b)^2+(a-b)^2}{(a-b)(a+b)}\div\frac{a+b-a+b}{(a-b)(a+b)}$$

$$\equiv\frac{a}{a-b}\times\frac{2a^2+2b^2}{(a-b)(a+b)}\div\frac{2b}{(a-b)(a+b)}$$

$$\equiv\frac{a}{a-b}\times\frac{2(a^2+b^2)}{(a-b)(a+b)}\times\frac{(a-b)(a+b)}{2b}\equiv\frac{?}{?}\text{ Ans.}$$

Explain and complete. Check by numerical substitution.

4. $$\frac{a^2+4a+3}{a^2-8a+7}\div\frac{a^2+8a+15}{a^2-9a+8}\times\frac{a^2-2a-35}{a^2-7a-8}\equiv ?$$

Try Exercise 49, page 436.

Complex Fractions

1. Simplify:

$$\frac{x - \frac{1}{x}}{1 + \frac{1}{x}}$$

Plan of attack: Multiply numerator and denominator by x.

Check: Let $x = 2$

In the original fraction

$$\frac{2 - \frac{1}{2}}{1 + \frac{1}{2}} \equiv \frac{1\frac{1}{2}}{1\frac{1}{2}}$$

In the answer

$$2 - 1$$

$$1$$

2. Simplify:

$$\frac{\frac{x}{y^2} + \frac{y}{x^2}}{\frac{1}{x^2} - \frac{1}{xy} + \frac{1}{y^2}}$$

Plan of attack: Multiply numerator and denominator by x^2y^2. Complete and check.

3. Simplify:

$$\frac{a + \frac{1}{a+b}}{a - \frac{1}{a-b}}$$

Plan of attack:

$$\frac{a + \frac{1}{a+b}}{a - \frac{1}{a-b}} \times \frac{(a+b)(a-b)}{(a+b)(a-b)} \equiv \frac{a(a+b)(a-b) + a - b}{a(a+b)(a-b) - (a+b)}$$

$$\equiv \frac{(a^2 + ab + 1)(a - b)}{(a^2 - ab - 1)(a + b)}. \quad \text{Ans.}$$

4. Contrast the two following fractions. Simplify the first

by multiplying numerator and denominator by 2, and simplify the second by multiplying numerator and denominator by 3.

$$\frac{\frac{1}{2}}{3} \qquad \frac{1}{\frac{2}{3}}$$

5. Contrast the three following fractions. Simplify the first by multiplying numerator and denominator by 4. In the second fraction, simplify first the fraction which contains the digits 2, 3, and 4. In the third fraction, simplify first the fraction which contains the digits 1, 2, and 3.

$$\frac{\frac{1}{2}}{\frac{3}{4}} \qquad \frac{1}{\frac{\frac{2}{3}}{4}} \qquad \frac{\frac{\frac{1}{2}}{3}}{4}$$

Simplify:

6. $\dfrac{x+2-\dfrac{3}{x}}{x-3+\dfrac{2}{x}}$

7. $\dfrac{\dfrac{1+c}{c}}{1-\dfrac{1}{c^2}}$

8. $\dfrac{1+\dfrac{a}{a+b}}{1-\dfrac{a}{a+b}}$

9. $\dfrac{\dfrac{2}{y}-\dfrac{1}{x+y}+\dfrac{1}{x-y}}{\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}}$

Try Exercise 50, page 438.

Tests

Test A

Insert $\equiv$ or $\not\equiv$ between each two of the following expressions:

1. $\dfrac{1}{(x+1)}-1 \qquad \dfrac{-x}{(x+1)}$

2. $\dfrac{-m-(2-m)}{m} \qquad \dfrac{-2}{m}$

3. $\dfrac{(r-1)^2}{2}+r-1 \qquad \dfrac{(r-1)^3}{2}$

4. $\dfrac{(a-b)(d-c)(r-s)}{(b-a)(c-d)} \qquad r-s$

Perform the operations indicated and check the results:

5. $\dfrac{1}{a+2}+\dfrac{2}{a-2}-\dfrac{2a}{a^2-4}$ 6. $\dfrac{2a}{a^2-1}-\dfrac{2a-1}{a-1}-\dfrac{a^2+2}{(a+1)^2}$

7. $\left(2+\dfrac{2a}{a-b}\right)\left(1-\dfrac{a-b}{a+b}\right)$

8. Find the reciprocal of 6 to the nearest thousandth. Verify your result by the use of logarithms.

Test B

1. Express the sum of $\dfrac{5}{(r-2)}$ and $\dfrac{6}{(r-3)}$ as a single fraction.

2. Is $\dfrac{3x^2}{a^2+a+1}$ equal to $\dfrac{3x^2}{a^2}+\dfrac{3x^2}{a}+\dfrac{3x^2}{1}$?

3. Is $\dfrac{3x^2}{a^2+a+1}$ equal to $\dfrac{3}{a^2+a+1}+\dfrac{x^2}{a^2+a+1}$?

4. Is $\dfrac{3x^2}{a^2+a+1}$ equal to $\dfrac{x^2}{a^2+a+1}+\dfrac{x^2}{a^2+a+1}+\dfrac{x^2}{a^2+a+1}$?

*5. Simplify: $$\frac{\dfrac{x}{y}-\dfrac{y}{x}}{\dfrac{x}{y}+\dfrac{y}{x}-1}-\frac{1+\dfrac{y^2}{x^2}-\dfrac{y}{x}}{\dfrac{x}{y}+\dfrac{y^2}{x^2}}.$$

6. Give the three meanings of a fraction.

Part II. Equations Containing Fractions

Proportion

1. What a proportion is. An equation in the form $\frac{a}{b}=\frac{c}{d}$ is sometimes called a proportion, and in it a and b are said to be proportional to c and d. A proportion is a statement of the

equality of two fractions; and since a fraction may be defined as a ratio, a proportion may be defined as a statement of equality between two ratios. When this is done, it is to be understood that the ratios are expressed as common fractions. The fact that equations of this kind have been designated by a name of their own indicates that they have been considered particularly important.

Which of the following equations are proportions? How do you know?

① $\frac{a}{b} = \frac{c}{d}$ ② $\frac{w}{x} = \frac{y}{z} + 2$ ③ $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

④ $\frac{b}{a} = \frac{d}{c}$ ⑤ $\frac{a}{c} = \frac{b}{d}$ ⑥ $\frac{a+b}{b} = \frac{c+d}{d}$

⑦ $\frac{a-b}{b} = \frac{c-d}{d}$ ⑧ $\dfrac{\frac{a}{b} - \frac{c}{d}}{\frac{a}{b}} = 0$ ⑨ $\frac{x}{y} + 3 = \frac{w}{z+4}$

⑩ $ad = bc$

2. Transformations of a proportion. Transform equation ② above into the proportion form by rewriting the right hand member as one fraction. Transform equation ⑩ into the form of a proportion by dividing by bd.

3. Show that equation ③ may be transformed so as to be identical with equation ①.

Plan of attack:

① $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

② $(a+b)(c-d) = (a-b)(c+d)$ ① × $(a-b)(c-d)$

③

④ $2\,bc = 2\,ad$

⑤ ④ ÷ ? Select a divisor which will give the required denominators.

4. Transform $\frac{a}{b} = \frac{c}{d}$ into the form $\frac{b}{a} = \frac{d}{c}$, and also into the form $\frac{a}{c} = \frac{b}{d}$.

5. Transform into the form $\frac{a}{b} = \frac{c}{d}$ each of the following equations of Exercise 1: ⑩, ⑥, ⑦, ⑧.

When proportions were used in mathematics before the modern symbolism of algebra was developed, it was customary to give verbal names to the parts of these equations and to the transformations. For instance, the first numerator and the second denominator were called the "extremes," the other terms were called the "means"; equation ⑩ was obtained from equation ① by a law that said, "The product of the means equals the product of the extremes." Equation ④ was obtained from equation ① by "inversion"; equation ⑤ by "alternation"; equation ⑥ by "composition" or "addition"; equation ⑦ by "division" or by "subtraction"; and equation ③ by "composition and division." Solving equation ① for d was called "finding a fourth proportional to a, b, and c." Solving $\frac{a}{b} = \frac{b}{c}$ for b was called "finding a mean proportional between a and c"; solving it for c was "finding a third proportional to a and b." The symbolism of modern algebra has made nearly all of this verbal algebra obsolete, except the name *mean proportional*, concerning which see page 154. It is interesting to notice how long it has taken to bring about this replacement.

Try Exercise 51, A, B, page 440.

Some uses for proportion. Solve each problem below by setting up a proportion and substituting in it the known values.

6. The volumes of circular cylinders of the same height are proportional to their bases, and to the squares of their radii, and to the squares of their diameters. If a cylindrical silo 10 feet in diameter holds 1600 cubic feet, how much will a silo of the same height but 15 feet in diameter hold? Suggestion $\frac{V}{V_1} = \frac{d^2}{{d_1}^2}$.

7. A certain pine log 15 inches in diameter weighs 1200 pounds. Find the weight of another pine log of the same length but 2 feet in diameter.

8. The volume of a sphere may be found by the formula ① $V = \frac{4}{3}\pi r^3$, and the volume of another sphere by the formula ② $V_1 = \frac{4}{3}\pi r_1^3$. Dividing ① by ② gives $\frac{V}{V_1} = \frac{r^3}{r_1^3}$. This means that the volumes of the spheres are proportional to the —— of their radii. A spherical ball with a 10.0″ radius weighs 236 pounds. Find the weight of a similar ball with an 18.0″ radius. Answer to three-figure accuracy.

***9.** The weight of an object is inversely proportional to the square of its distance from the center of the earth. Find the weight of an aviator 5 miles above the surface of the earth if his weight at the surface is 170 pounds. Consider the radius of the earth as 4000 miles and answer to the nearest 10th of a pound. If the aviator could rise 1000 miles above the earth's surface, what would be his weight?

Writing Proportions in the *k* Form.

1. The sides of a triangle are $a = 18''$, $b = 14''$, $c = 11''$, and the largest side of a similar triangle $a_1b_1c_1$ is $15''$. Find the other two sides.

Plans of solution:

1. By proportion. Set up the proportion $\frac{a}{a_1} = \frac{b}{b_1}$ and solve as suggested in the examples above.

2. By an equation in the "*k* form." Write the equation $b_1 = \frac{15}{18}b$, or $b_1 = \frac{5}{6}b$, substitute and solve. The $\frac{5}{6}$ is the ratio of similarity; it is a constant for these two triangles. Representing this constant by k, we have $b_1 = kb$ for the general form of the equation.

Study the preceding two plans of solution until you understand them thoroughly.

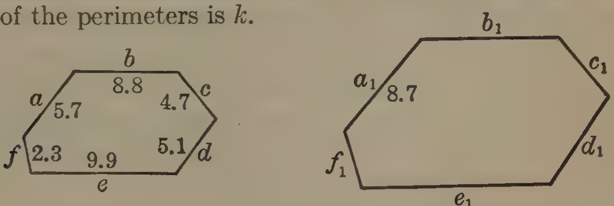
2. Consider the rectangle ab, in which $a = 5$ and $b = 12$. Evidently $b = \frac{12}{5}a$, that is, $k = \frac{12}{5}$. If the rectangle changes size without changing shape, and its new dimensions are designated by a_1 and b_1, then $b_1 = \frac{12}{5}a_1$. As the rectangle continues to change in size without changing in shape, k remains constant. If $a_1 = 17$, find b_1. First plan: $\frac{a}{b} = \frac{a_1}{b_1}$, etc. Second plan: $b_1 = kb$, or $b_1 = \frac{12}{5} \times 17$.

Supply the missing numbers:

a_1	32		.08	
b_1		41		9.72

3. The sides of a polygon are 8, 12, 9, and 16. The largest side of a similar polygon is 25. Find its other sides. Express the results as mixed numbers.

***4.** In the two similar polygons illustrated here, corresponding sides are proportional. Write an equation in the k form involving a_1 and a. Find k; then find the numbers missing in the larger figure. (Work to three-figure accuracy and round off each result to two-figure accuracy.) Check by showing that the ratio of the perimeters is k.



5. In the continued proportion $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h}$, prove that $\frac{a+c+e}{b+d+f} = \frac{g}{h}$. *Plan of proof:* Let ① $\frac{a}{b} = k$, then ② $\frac{c}{d} = k$,

and ③ $\dfrac{e}{f} = k$; therefore, ④ $a = bk$, ⑤ $c = dk$, ⑥ $e = fk$,

and ⑦ $a + c + e = (b + d + f)k$, ④ + ⑤ + ⑥

⑧ $\dfrac{a + c + e}{b + d + f} = k$, etc. ⑦ ÷ $(b + d + f)$

*6. The length of a pendulum varies as the square of the time of a complete vibration. This relation may be expressed in the form $\dfrac{l}{l_1} = \dfrac{t^2}{t_1^{\,2}}$, or in the form $l = kt^2$. When $t = 1$ second, $l = 24.8''$, and when $t = 2$ seconds, $l = 99.3''$. Show that $k = 24.8$. Find the length of a pendulum which will vibrate in 3 seconds, in 4 seconds, in 5 seconds, in 10 seconds. Find the time of vibration of an eight-foot pendulum. Work to three-figure accuracy.

Try Exercise 51, C, page 441.

Other Fractional Equations

Many of the equations in the following exercise and the accompanying drills have been prearranged so that a knowledge of factoring is helpful in solving them. In many of them there has been a multiplication by -1 so as to change such an expression as $a - b$ into the form $b - a$. In some solutions time can be saved by simplifying one or both members before multiplying by the common denominator. When both monomial and binomial denominators are involved, it may be easier to remove the monomial denominators first. Re-read paragraph 8 of *How to Study Algebra* (see page xx).

1. Solve and check:

① $\dfrac{5x^2 + 6}{x^2 - 4} = 5 - \dfrac{3}{2 - x} - \dfrac{7}{x + 2}$

② $\dfrac{5x^2 + 6}{(x + 2)(x - 2)} = 5 + \dfrac{3}{x - 2} - \dfrac{7}{x + 2}$ Explain.

③ $5x^2 + 6 = 5(x^2 - 4) + 3x + 6 - 7x + 14.$ ② × ? Complete.

Check: $\dfrac{5(-\frac{3}{2})^2 + 6}{(-\frac{3}{2})^2 - 4}$ $\qquad 5 - \dfrac{3}{2 - (-\frac{3}{2})} - \dfrac{7}{-\frac{3}{2} + 2}$

$\dfrac{\frac{45}{4} + 6}{\frac{9}{4} - 4}$ $\qquad 5 - \dfrac{3}{2 + \frac{3}{2}} - \dfrac{7}{\frac{1}{2}}$

$\dfrac{\frac{69}{4}}{-\frac{7}{4}}$ $\qquad 5 - \frac{6}{7} - 14$

Explain and complete. Remember that in checking, the right and left members are not to be changed in value.

2. Solve and check:

$$\frac{5}{x} + \frac{6}{y} = 7 \qquad \frac{7}{x} + \frac{9}{y} = 10$$

Plan of attack: Multiply the first equation by 3 and the second by 2, then subtract the fourth equation from the third. Another plan is to clear of fractions at the outset.

***3.** Solve and check:

$$\frac{3x + y}{y - x} + 2 = \frac{12}{x - y} \qquad \frac{x + 7}{y} + \frac{17 + x}{5 - x} = 3$$

4. Solve, and check both roots:

① $\dfrac{x^2 + 10x + 53}{x^2 - 4} = \dfrac{2x + 1}{x + 2} + \dfrac{4x + 5}{2 - x}$

② $\dfrac{x^2 + 10x + 53}{x^2 - 4} = \dfrac{2x + 1}{x + 2} - \dfrac{?}{?}$

5. Solve for x and check:

① $\dfrac{r}{x - r} - \dfrac{s}{x - s} = \dfrac{r^2 - s^2}{s(x - s)}$

② $rs(x - s) - s^2(x - r) = (r^2 - s^2)(x - r)$ ① × ?

③ $rsx - r^2x = rs^2 - r^3$

Complete by factoring each member, etc.

Try Exercise 52, page 445.

Verbal Problems

Read the "Plan of Problem Solution," page 169.

1. The sum of $\frac{1}{3}$ of a certain number, $\frac{2}{5}$ of it, and $\frac{1}{9}$ of it is $5\frac{6}{7}$ more than $\frac{5}{7}$ of it. Find the number.

2. A, B, and C together have \$1285. A's share is \$25 more than $\frac{5}{6}$ of B's, and C's share is $\frac{4}{15}$ of B's. Find the share of each.

3. A man sold two acres more than $\frac{3}{5}$ of his farm and had four acres less than half of it left. Find the number of acres in the farm.

4. One number is $\frac{2}{3}$ another, and their product plus their sum is 69. What are the numbers?

5. The sum of two fractions of which the numerators are 3 is three times the smaller; and three times the smaller subtracted from twice the larger is $\frac{3}{8}$. What are the fractions?

6. If 2 is added to the numerator of a certain fraction, the sum of the given fraction and the new fraction is $1\frac{1}{2}$. If 2 is subtracted from the numerator of the original fraction, the sum of the original fraction and the new fraction is 1. Find the original fraction.

7. A young man invested part of \$1000 at 4% and the rest at 5%. His total income was \$48. How was the money divided?

8. What sum must a man invest in order to get an annual income of \$176 if $\frac{4}{5}$ of the sum earns $4\frac{1}{2}$% and the rest 4%?

9. A man has \$3000 at interest. On \$500 he receives 5%, on part of the rest 3%, and on the rest 4%. The total interest for 3 years was \$330. How much was invested at 3%?

10. A printing press did a certain piece of work in 4 hours; when a second press was installed, a similar job was completed by both presses in $2\frac{1}{2}$ hours. On the following day the second press had to do the work alone. How many hours should be allowed?

11. The denominator of a certain fraction is 7 less than the numerator. If 5 be added to the numerator, the value of the fraction is $\frac{9}{5}$. What is the fraction?

12. Divide the number c into two parts such that a times the larger shall be b times the smaller.

***13.** Find two numbers, of which the sum is c, such that b times the first exceeds a times the second by d.

14. One train runs 84 miles in the same time in which a second train runs 96 miles. If the rate of the first is 3 miles an hour less than that of the second, what is the rate of each?

15. A boy rode his wheel to a certain place at the rate of $8\frac{1}{2}$ miles an hour and returned by a road $2\frac{1}{4}$ miles longer at the rate of 9 miles an hour in 10 minutes longer time. How long is each road?

16. A can lay a brick pavement in 8 days, and B in 10 days. In how many days can they lay it working together?

17. A broker bought a certain number of shares of stock for \$1200. After the price of each share had advanced \$40, he sold all but three of his shares for \$1440. How many shares did he buy?

18. If flour costs b cents a pound and sugar c cents a pound, and if a man buys twice as much flour as sugar, how many pounds of each can he buy for m dollars?

19. From two cities 32 miles apart, two men start at the same time to meet each other, one going 3 miles an hour and the other 5 miles an hour. How long will it be before they meet, and how far will each have gone?

***20.** From two cities a miles apart, two men start at the same time to meet each other, one going m miles an hour and the other n miles an hour. How long will it be before they meet, and how far will each have gone?

21. \$1.20 is to be divided among three children in such a way that their shares shall be proportional to their ages. The sum of their ages is 24 years, and each child is 2 years older than the next younger child. How much will each receive?

Tests

Test A

1. What are the three meanings of a fraction?

Insert $\equiv$ or $\not\equiv$ between each two expressions:

2. $a - \dfrac{a^2+b^2}{a} \quad b^2$

3. $\frac{2}{3}+\frac{1}{4} \quad \frac{3}{7}$

4. $5+\dfrac{3}{x} \quad 5x+3$

5. $\frac{2}{5}\cdot 3 \quad \frac{6}{5}$

6. $\dfrac{(m-1)}{2}\cdot(m-1) \quad \dfrac{(m-1)}{2(m-1)}$

7. $\dfrac{b-a}{b^2-a^2} \quad \dfrac{1}{a+b}$

Perform the operations indicated:

8. $\dfrac{2m+1}{2m-1}+\dfrac{8}{1-2m}-\dfrac{2m-1}{2m+1}$

9. $\dfrac{b}{s}+\dfrac{cs-br}{rs+s^2x}$

10. $\dfrac{\dfrac{a}{a-3}-\dfrac{a}{a+3}}{\dfrac{a+2}{a-3}-\dfrac{a-2}{a+3}}$

***11.** $\dfrac{\dfrac{m-3}{3r}-\dfrac{3r}{m-3}}{\dfrac{1}{3r}-\dfrac{1}{m-3}}$

Simplify by multiplying numerator and denominator by the same number:

12. $\dfrac{5\frac{1}{2}}{4\frac{2}{3}}$

13. $\dfrac{x-1}{x+\frac{2}{5}}$

14. $\dfrac{-3}{y-\dfrac{x}{2}}$

15. $\dfrac{x-\dfrac{x}{y}}{y-\dfrac{x}{y}}$

16. $\dfrac{1}{\sqrt{2}}$

17. $\dfrac{\sqrt{5}-3}{\sqrt{3}-\sqrt{5}}$

18. $\dfrac{xy^{-1}}{z^{-3}}$

19. $\dfrac{a}{x^{-1}-y^{-2}}$

Test B

Reduce to lowest terms:

1. $\dfrac{x^2 - 2xy - z^2 + y^2}{3x^2 - 3xy + 3xz}$

2. Unite: $\dfrac{a-b}{a^2 + 3ab + 2b^2} - \dfrac{a+b}{a^2 + ab - 2b^2}$

Perform the operations indicated:

3. $(b^2 - 4)\left(\dfrac{b}{b+2} - \dfrac{1-b}{2-b}\right)$ *4. $\left(\dfrac{x}{y} - \dfrac{y}{x}\right) \div \left(\dfrac{3x^2 + y^2}{x^2 - xy} - 2\right)$

Write in simplest form:

5. $\dfrac{a^2\sqrt{7} - 2a\sqrt{6}}{a^2 + 5a}$ *6. $\dfrac{\dfrac{s^3}{r^3 + s^3} - \dfrac{s}{r+s}}{1 - \dfrac{2s^3}{r^3 + s^3}}$

Solve and check:

7. $\dfrac{1}{2(3y+7)} - \dfrac{2}{3y^2 + 22y + 35} + \dfrac{1}{2y+10} = 0$

8. $3 + \dfrac{4x - 3x_1}{8} = 7 - \dfrac{x_1 - 28x}{4},$

$9 - \dfrac{4x - 5x_1}{7} = 10 + \dfrac{6x + 2x_1}{3}$

9. $\dfrac{x+8}{x+3} + \dfrac{x(x+5)}{x^2 - 9} = -\dfrac{7+x}{3-x}$

10. $\dfrac{a}{b+x} = \dfrac{b}{a-y}, \quad \dfrac{a}{b+y} = \dfrac{b}{a-x}$

11. $\dfrac{4}{x - 4b} - \dfrac{1}{x+b} = \dfrac{4}{x+4b} - \dfrac{1}{x+3b}$

12. The resistance of a certain electrical circuit varies inversely as the square of the diameter of the wire. Write a literal proportion to illustrate this relation. If the resistance is 2.5 units when the diameter of the wire is 0.015″, what is the resistance if wire 0.12″ in diameter is used?

13. Two arithmetic progressions begin with the same number, namely 4, and the difference in the second is $\frac{1}{2}$ the difference in the first. The sum of the first seven terms of the second is $\frac{5}{8}$ the sum of the first seven terms of the first. Write the first three terms of each progression.

Test C. Review

Solve by formula:

1. $2a^2x^2 + 3ax + 1 = 0$

***2.** $\dfrac{kl - mn^2 + k}{lm} = \dfrac{lm - m}{2kl}$ (solve for k)

3. If $v = \frac{1}{3}Bh$, $B = a^2$, and $h^2 = \left(\dfrac{t - a^2}{2a}\right)^2 - \left(\dfrac{a}{2}\right)^2$, find v in terms of a and t.

4. If $h^2 = b^2 - c^2$, $c = \dfrac{a}{2}$, and $b = \dfrac{t - a^2}{2a}$, find h in terms of a and t.

5. (*a*) $\dfrac{a^{-2}}{b^{-3}} \times \dfrac{a^2b^3}{a^2b^3} = ?$ (*b*) $\dfrac{a^{-2} + b^{-3}}{a^{-3} - b^{-2}} \times \dfrac{a^3b^3}{a^3b^3} = ?$

6. Find the value of $\dfrac{497.6\sqrt{85.73}}{\sqrt[3]{0.1462}}$

***7.** A man deposited $2500 in the bank on the day his son was born. To what did this amount when his son was 21, if interest at 4% was compounded annually?

8. A man is to walk for 30 minutes at $3\frac{1}{2}$ miles an hour up a slope which rises at an angle of $8° 20'$ with the horizontal plane. How high will he then be above the horizontal plane from which he started?

9. The sides of a triangle are $5.000'$, $12.00'$, and $13.00'$. Find the smallest angle.

***10.** Two automobiles start at the same time from cities 180 miles apart and meet at the end of four hours. Repeating the trip on the following day, the first car travels twice as fast as before and the second car three fourths as fast as before, and they meet in three hours. What was the average rate of each car on the first trip?

11. When an airplane is gliding with its power off, the "gliding angle" is the ratio of the number of feet the plane moves forward (horizontally) to the number of feet it drops in the same time. When this ratio is $\frac{3}{2}$, how far will the plane travel horizontally in dropping one mile? What angle does its course make with the horizontal?

CHAPTER VIII

DEPENDENCE

TYPES OF DEPENDENCE. METHODS OF STUDYING DEPENDENCE. VARIATION. GRAPHS. STATISTICS

Introduction

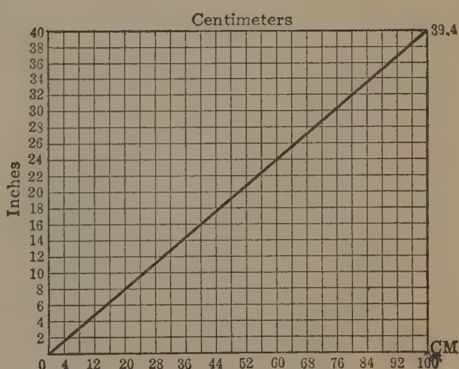
1. What are the three central ideas of this course? (If necessary, refer to page 1.) With which of the three do you think the present chapter is chiefly concerned?

2. In order to understand something of the importance of the idea under consideration, read the following list of studies of dependent variables and add to it: Doctors study changes in bodily temperature and changes in the rate at which wounds heal. Insurance experts study changes in the expectation of life as the age of the insured person varies. Geologists study the rate at which rivers (the Niagara, for example) wear away the rocks over which they flow. Notice also that when the relationship between two variables is understood and can be mathematically expressed, problems about these variables can be solved and sometimes their future course predicted, just as your life expectancy if you are living in 1950 can be predicted, or the coming of an eclipse can be predicted by an astronomer.

***3.** More and more of the sciences are becoming exact, that is, mathematical. This is true, for instance, of parts of economics and biology. Can you mention others? More and more it is necessary to understand the methods of mathematics in order to understand what man is discovering about the world.

c	i
1	0.394
5	
10	3.94
20	
30	
70	
100	39.4

4. Illustrative studies. In the accompanying table, c equals the number of centimeters in a certain distance and i the number of inches in the same distance. Supply the missing numbers. Explain the formula $i = 0.394\,c$. How do you know that $c = 0.394\,i$ is not the correct formula? Is a given number of centimeters larger or smaller than the corresponding number of inches? In the formula $c = ? \times i$, supply the missing coefficient. The graph below was constructed with the help of the preceding table. (*a*) Explain the horizontal scale, the vertical scale. (*b*) Show that 20 inches is approximately 51 centimeters. (Follow the horizontal line marked 20 to its intersection with the graph and find on the horizontal scale the reading which is directly below the intersection.) (*c*) Read from the graph the approximate number of centimeters in 1 in., 7 in., 30 in., 26 in., 9 in., 16 in. (*d*) Read from the graph the approximate number of inches in 10 cm., 28 cm., 52 cm., 82 cm. (*e*) What is the number of centimeters in 150 in.? (Find the number of centimeters in 15 in. and multiply it by 10.) In 200 cm.? 230 cm.? 1200 cm.? 1000 cm.? It is evident from the preceding exercises that we can interpolate on the graph. Explain. (*f*) The graph is a straight line and can be indefinitely extended upward to the right, hence we can *extrapolate*; that is, we can find values beyond those shown on the part of the graph

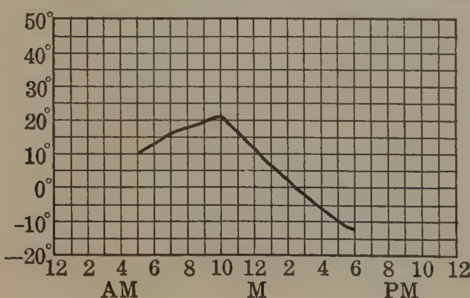


which we have drawn. If the graph were extended to 500 inches, what corresponding number of centimeters would it show? (*g*) By means of a formula of this study, check some of your answers for (*c*) and (*d*). Comment on the relative speed of the formula method and the graphical method for finding these numbers; on the relative accuracy. Could we increase the accuracy of the graphical method by drawing an accurate graph on a larger scale? (*h*) Is the formula used in this study in the form of a *linear* equation in two variables? (*i*) Explain the statement that in studying the formula $i = 0.394\ c$ in which i represents a number of inches, we have thought of one letter as changing in value and then considered resulting changes in the value of the other letter which is called the "dependent variable." This kind of thinking has proved valuable in mathematics. It has added much to man's interest in and understanding of the world. Try to make use of it in your own thinking.

5. On a winter's day in Maine, temperature readings were as follows:

Time	A.M. 6	8	10	12 M.	2	4	6	8 P.M.
Temperature in degrees	10	17	21	8	2	− 6	− 12	?

From these data we obtain the accompanying graph. (*a*) What was probably the approximate temperature at 7 A.M.? at 11 A.M.? 3 P.M.? 5 P.M.? Can you interpolate on this graph? (*b*) Approximately what was the probable temperature at 8 P.M.? (*c*) Can you tell with the help of the graph the approximate temperature

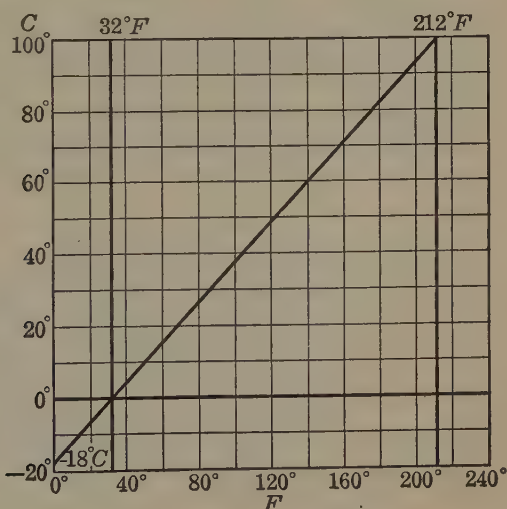


at 6 o'clock the following morning? At 8 o'clock two mornings later? That is, can you extrapolate as you did in using the graph of the preceding study? Why not? (*d*) Can you write a formula which will enable you to read the temperature in your town for any given time in the future? In this particular, contrast this study with the one preceding.

6. Many thermometers give readings in both Fahrenheit and Centigrade scales. The centigrade reading which corresponds to any given Fahrenheit reading can be found by subtracting 32 from the Fahrenheit reading and multiplying the result by $\frac{5}{9}$. (*a*) Express this relation in a formula, using the letters F and C. (*b*) With the help of the formula supply the numbers missing in the following table, writing them to the nearest degree:

F.	0		– 6			64
C.		100		– 15	– 30	

(*c*) With the help of the table above, explain and verify the graph below.



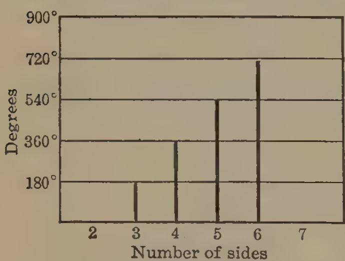
(*d*) Read from the graph the approximate values of the missing numbers:

F.	18°		212°			$-18°$
C.		40°			$-10°$	

(*e*) Can you extrapolate here? Give a reason for your answer. Indicate if you can by means of the corresponding values of F and C two points outside the diagram which would be on the graph if it were sufficiently extended.

(*f*) Is the formula of this study in the form of a linear equation in two variables? How does the graph bear out your answer?

7. In any polygon the sum of the interior angles is equal to 180° taken as many times less two as the polygon has sides. Express this fact as a formula, using i for the sum of the interior angles and n for the number of sides. Explain the accompanying graph. Make a table of values of i and n and tell how to complete the graph. Can we represent the facts of the formula by a line through the tops of the bars? Can we find a value on the vertical scale to correspond to $3\frac{1}{2}$ on the horizontal scale? Explain. Contrast with example 5 above.



Four Methods of Representing Dependence

1. Just as in the use of language there are many ways of expressing one idea, so in mathematics there are many ways of expressing the dependence of one number upon another. The four standard methods which are used in this course are all illustrated in the studies above. They are: (I) By verbal description. (II) By formula. (III) By table. (IV) By graph. Illustrate each method.

Of the four methods of representing dependence, the graphic is the only one which needs comment here. Notice that each graph in the preceding studies has its scales on two intersecting axes. The scales on the two axes may together be called the *Cartesian scale* after Descartes who lived at about the time the Pilgrims landed in America and who invented this method of discussing pairs of mutually dependent variables. The Cartesian scale is a scale for representing pairs of numbers. By its use any point on a plane can be represented by two numbers (sometimes called the "Cartesian coördinates" of the point) one of which gives the distance to the right (positive) or left (negative) from the vertical axis, and the other the distance above (positive) or below (negative) the horizontal axis. Places in cities which are laid out in square blocks are located in a similar way when we say, for instance, that A lives four blocks east and two blocks north of B. Latitude and longitude are measured in a similar way from the axes formed by the equator and the zero meridian. In this course it will be convenient to restrict ourselves to axes which are mutually perpendicular. If necessary, try **Exercise 53, A, B, C,** page 448.

2. The circumference of a circle is approximately 6.28 times its radius. Represent this relation (1) by a formula or equation, and (2) by a table. Extend the values from $r = 0$ to $r = 10$. (3) By a graph.

3. A series of lamps of various sizes is so priced that each lamp, l, costs 50 cents more than twice as much as its shade, s. Represent this relation by a formula, by a table (from $s = 0$ to $s = \$15$), and by a graph.

4. The plate for printing a certain map of the United States cost \$25 and the printing costs 3 cents per copy. Represent the total cost, c, of x copies, by a verbal statement, by a formula, by a table extending to $x = 2000$, and by a graph.

5. The formula for the area of a circle is $A = \pi r^2$. Represent this relation in the three other methods of the preceding exercise. Notice that the graph is not a straight line: the equation is *not linear*.

*6. For the formula for the volume of a sphere make a study similar to that of the preceding exercise.

7. Sum up the purposes served by each of the four methods of representing dependence. Notice that a formula is a generalization; it is a concise method of stating the law of the dependence in algebraic symbols, and it is useful in solving problems. Notice that a graph pictures the nature of the dependence and shows facts at a glance. The great importance of the graphical method can only be hinted at in this chapter, but it is true that this method opened the way for many mathematical discoveries and that it enabled mathematicians to understand and explain matters which were otherwise out of their reach. It is also true that graphs are much used in popular discussions.

Try Exercise 53, D, page 449.

Part I. Types of Dependence

Proportional Variation

1. Any equation in two variables which can be expressed in two terms, each of which contains one variable, illustrates the type of dependence called *proportional variation*. In the following illustrations k represents a constant and x and y represent variables.

(*a*) $y = kx$ — y varies directly as x, or y is proportional to x. If x is doubled, then y is doubled, etc.

(*b*) $y = kx^2$ — y varies as the square of x, or y is proportional to the square of x. If x is doubled, then y is increased fourfold, etc.

(*c*) $y = kx^3$ y varies as the cube of x.

(*d*) $y = kx^{\frac{1}{2}}$ y varies as the square root of x. (It is evident that "varies as" means *is equal to a constant times* . . .)

(*e*) $y = \frac{k}{x}$ y varies inversely as x, or y is inversely proportional to x. If x increases, y decreases.

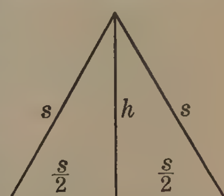
(*f*) $y = \frac{k}{x^2}$ y varies inversely as the square of x.

In $d = rt$, d varies directly as r and directly as t; that is d varies jointly as r and t. If d is a constant, then r varies inversely as t. Explain. In $y = \frac{kx}{wv^2}$, y varies directly as x, inversely as w, and inversely as the square of v. Notice that all the examples in proportion, page 229 ff., Chapter VII, are illustrations of proportional variation.

The expression *x varies with y* is used in a more general way to include the variation, not necessarily proportional, in any equation in which a change in x involves a corresponding change in y. It includes such equations as $x = ay + b$ and $x = ay^2 + by + c$.

Draw a graph for each type of variation 1 to 6 above if you have not already done so.

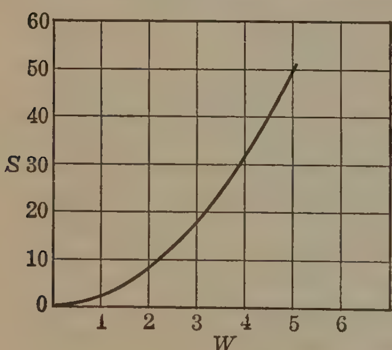
2. Illustrative study. In an equilateral triangle the altitude, h, varies directly as the side, s, and the area, A, varies as the square of the side. Express both of these relations in the "k form" and find the value of k in each one. Use these two formulas in supplying the numbers missing in the table on page 250. Write two proportions, one involving s, s_1, h, and h_1; the other involving s, s_1, A, and A_1. Use these proportions in checking the table. Contrast the k form with the



s	h	A
3		
7		
20		
	$8\sqrt{3}$	
	21	
		510
		318

proportion form. While a proportion may be convenient to use in solving some problems, mathematicians usually prefer the k form. A proportion expresses the relation between *two* pairs of values; the k form expresses the relation between the variables for *any* values. A proportion suggests fixed values; the k form suggests variables changing so as to cover the whole range of values, and mathematicians have found that it pays to think in this way. Keep this fact in mind as you make the following study.

3. Illustrative study. In a small model airplane the width is 2 ft. and the wing surface is 8 sq. ft. We are to study the dependence of values of width (w) and wing surface (S) as the model increases in size without changing shape. The geometric relation is that areas of similar figures are proportional to the squares of corresponding lines. Express this relation as a proportion. Extend the following table to $w = 10$. Which increases more rapidly, the wing surface or the width? How is this shown by the graph? Discuss the selection of the scales.



w	S
1	2
2	8
3	18
4	32
5	50
6	

From the table show that if $S = kw^2$, then $k = 2$. Verify this result by transforming the proportion $\frac{S}{S_1} = \frac{w^2}{w_1^2}$ into the form $S = (\ \ )w^2$ and substituting the given values for S_1 and w_1. In the equation $w^2 = kS$ find the value of k.

Three special methods of verbal description. We can say that the wing surfaces *are proportional to, have the same ratio as,* or *vary as, the squares of the width.* These three methods of verbal description apply only to that type of dependence called proportional variation. Notice that it is also true that the wing surface is equal to a constant times the square of the width.†

w	S
1	3
2	12
3	27
4	48

4. With the help of the accompanying table, find the value of k in the equation $S = kw^2$.

5. If the width of an airplane is doubled without changing the shape of the airplane, what change is made in the wing surface? If the width is made five times as great? Seven times as great?

6. An airplane 12 ft. wide has 432 sq. ft. of wing surface. Find by use of a proportion the wing surface of a similar airplane 18 ft. wide. Find the value of k in the equation $S = kw^2$, and so verify your answer.

7. An airplane 25 ft. wide has 450 sq. ft. of lifting surface. What is the lifting surface of a similar plane 42 ft. wide?

8. The wing surfaces of two similar airplanes are 800 sq. ft. and 3200 sq. ft. respectively. What is the ratio of their widths?

9. A small airplane model is 2 ft. long and 5 ft. wide. Make a study of the relation between length and width similar to illustrative study on page 250.

10. An airplane model 6 ft. long weighs 108 pounds. Assume that the weight varies as the cube of the length, and make a study of the relation of weight and length similar to the study on page 250.

11. From the formula $A = \pi r^2$, for the area of a circle, write

† In more advanced discussions of dependence, the word *function* is found. Another way of saying that S is dependent on w is to say that S is a function of w. This is true in general when if any particular value for w is given, the corresponding value for S is thereby determined. It is true in particular when S is equal to any algebraic expression containing w.

a proportion. Express the relation between A and r using first the word "proportional," then the word "ratio," then "varies as." In the equation $A = kr^2$, what is the value of k? In the equation $r = kA^{\frac{1}{2}}$?

***12.** The weights of two spheres of the same material vary as the cubes of their diameters. If a 10 in. sphere weighs 50 pounds, what diameter must be used in constructing a 400-pound sphere of the same material? Find the value of k in the equation $W = kd^3$. Find the weight of a 12-in. sphere, a 30-in. sphere.

***13.** As an eagle changes in size without changing shape, his lifting surface increases as the square of his length, and his weight increases as the cube of his length. Show that if he continues to grow, he will become too heavy to fly.† If his length is doubled, what changes in lifting surface and in weight result?

14. A boat traveled a distance d miles at rate r in t hours. What would be the effect on d of doubling both r and t? Give a numerical illustration.

15. Solve for r the formula $V = \frac{4}{3}\pi r^3$, and describe the variation involved, using the words "varies as the cube root of."

16. Describe the variation indicated by the formula $a = bc^2e^3\sqrt{g}$. What is the effect on a of doubling b? c? e? g?

***17.** Show that a large animal has a better chance than a small one to survive when exposed to extreme cold. Assume that the two animals differ in size only and that the amount of heat lost varies as the surface while the amount of heat generated varies as the volume. Suppose one animal twice as large as the other, and then ten times as large, etc.

***18.** If the weight of an animal is proportional to its volume and the strength is proportional to the area of cross section of

† Read Chapter III, "On Being the Right Size," J. B. Haldane's *Possible Worlds*. (Harper's.)

muscle, explain why a grasshopper or a frog can jump proportionally farther than an athlete can. Explain also the estimate made by an eminent biologist that if an elephant were ten times his normal size (length, width, and height) his legs would have to be large enough to cover all the ground under him and some more.

***19.** Why is it that if a brick and the powder from another brick which has been ground up are thrown into the air to the same height, the brick will reach the ground much sooner than the powder? Why is a small animal less liable to injury by a fall of a given distance through the air than is a large animal?

Try Exercise 54, page 455.

Linear Dependence, $y = ax +$

1. Introduction. A linear equation is an equation of the first degree; its graph is a straight line. What is the significance of the word *linear*? Which types of variation of example 1, pages 248, 249, are linear? We have already made use of pairs of equations in two unknowns in the solution of verbal problems. Here we need to consider only their representation by table and by graph.

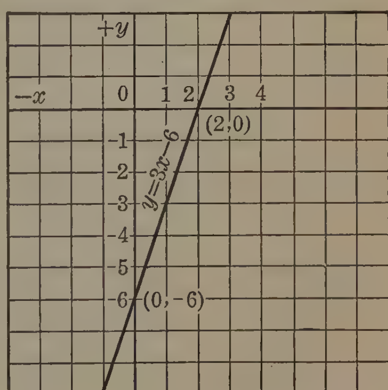
2. Plot the graph of

$$y = 3x - 6$$

Plan of procedure:

Supply what is missing in the working table:

x	$y = 3x - 6$	y
0	$y = 0 - 6$	-6
		0
5		
7		
-1		



Explain the construction of the graph.

(The proof of the fact that the graph of any equation of the first degree in two variables is a straight line is a simple matter, but it is not required in this course.) Where does the graph cross the x-axis? the y-axis? The distances from *origin* (the point of intersection of the axes) to these two points are called the *intercepts* of the graph. These are often the most convenient points to locate when plotting a linear graph. How many points are necessary to determine the location of a straight line? Explain the desirability of locating a third point. Explain why it is better in drawing the graph to locate two points at a reasonable distance apart rather than two points very close together.

The graph is called the *locus* of an equation because it is the *place* where a point must lie if its coördinates satisfy the equation.† How many pairs of numbers satisfy the equation we are studying? Let $y = 0$, and solve the resulting equation. How is the root of the equation $3x - 6 = 0$ indicated on the graph? In the same way discuss the case in which $x = 0$.

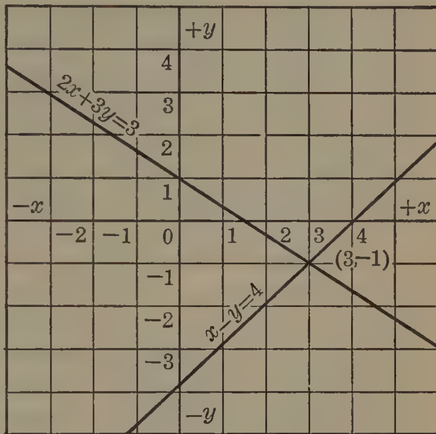
3. Plot the graph of $y = 5x + 10$. Discuss it as you did the graph of the preceding example. On the same axes plot the graph of $y = 5x + 15$. Observe that the two graphs are parallel.

Graphic Solution of the Linear Pair

1. Consistent and inconsistent equations. Plot the graph of $x + 2y = 11$, and on the same axes the graph of $x + y = 7$. What are the coördinates of the point of intersection of these graphs? Show that these numbers satisfy both of the equations. Tell how to solve graphically a pair of linear equations in two unknowns. Study the accompanying figure. It represents the graphical solution of the equations $2x + 3y = 3$ and

† In speaking strictly, the word *locus* must also imply that there are no points in the graph which do not satisfy the equation.

$x - y = 4$. Study the equations of Ex. 3, p. 254 and tell whether you can solve them graphically. These equations are "inconsistent." Explain.



2. Dependent equations. Represent the variation in $y = 3x - 5$ and in $x = \frac{(y+5)}{3}$ in two tables like that of example 2, page 253. From these tables can you draw two distinct graphs? Examine the original equations; are they distinct equations or are they equivalent? Explain the title of this paragraph. How do the foregoing graphs show that two linear equations in two unknowns cannot be solved unless they are consistent and independent?

3. Solve graphically $x + 2y = 12$ and $3x - y = 1$.

4. Plot on the same axes graphs of (1) $x + 3y = 4$ and (2) $2x - y = 1$, (3) $2x - y = 3$, (4) $2x = 8 - 6y$. Among these equations find a pair which are inconsistent; a pair which are dependent; the solution to a pair which are consistent and independent.

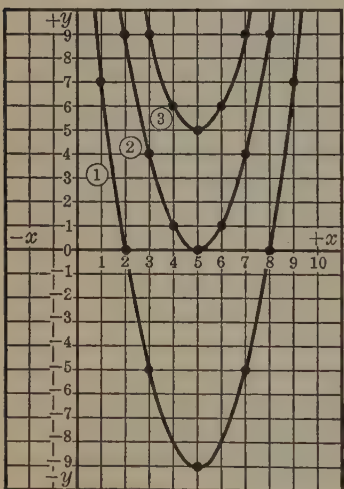
Graphs of equations are important, not so much because they provide another means of solving equations, as because they provide another method for studying the relations involved. This method has proved helpful to mathematicians and opened the door to many new mathematical ideas.

Try Exercise 53, E, page 451.

Quadratic Dependence

1. What is a quadratic equation in one unknown? Which type of variation, page 248, example 1, is quadratic? We have already made use of quadratic equations in one unknown in the solution of verbal problems. In Chapter X we shall use quadratic equations in two unknowns for the same purpose. Here we shall consider the representation of the quadratic equation of dependence by table and by graph.

2. Plot the graphs of the three equations ① $x^2 - 10x + 16 = y$, ② $x^2 - 10x + 25 = y$, and ③ $x^2 - 10x + 30 = y$. In making the tables of these values, select at least 8 values for x and find the corresponding values for y. Explain the construction of the graph. The roots of the equation $x^2 - 10x + 16 = 0$ are 2 and 8. Show how these roots are indicated on the graph. Tell how to solve a quadratic equation graphically. Discuss the graphic solution of $x^2 - 10x + 25 = 0$. Show by the formula that the equation $x^2 - 10x + 30 = 0$ has imaginary roots. The graph indicates imaginary roots by its failure to cross the x-axis.



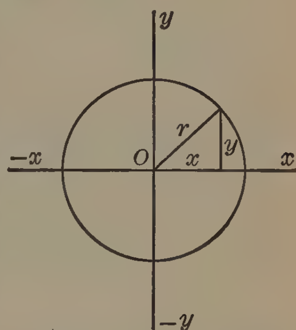
3. Solve graphically the equation $x^2 + x - 12 = 0$.

Plan of solution: Set the expression equal to y; make a table of values; plot the graph. In case any part of the graph is not clearly indicated by the values in the table, explore that part of the graph by finding additional values for x and y.

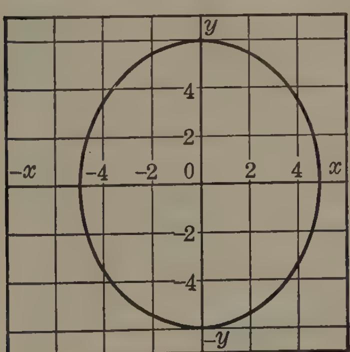
4. Solve graphically the equation $x^2 + 2x - 15 = 0$. Re-

place the -15 by a number that makes the expression a perfect square and draw the graph. Replace the -15 by a positive number so large that the roots are imaginary. Plot the graph. Tell what happens to the roots of the equation when the graph moves gradually from the lowest position to the highest. Each of the curves of examples 3 and 4 is a *parabola*. The parabola is used in making reflectors for searchlights, in whispering galleries, and in sound detectors such as those used in detecting the sound of distant airplane motors in the World War.

5. Plot the graph of $x^2 + y^2 = 25$. Study the right triangle in the illustration. Show that $x^2 + y^2 = r^2$ and hence the graph is a circle with its center at the origin and radius 5. Show that any value of either letter greater than 5 gives an imaginary value for the other value. This indicates the regions into which the graph cannot go.



6. Plot the graph of $x^2 + y^2 = 16$. What is a simple method of constructing this graph?

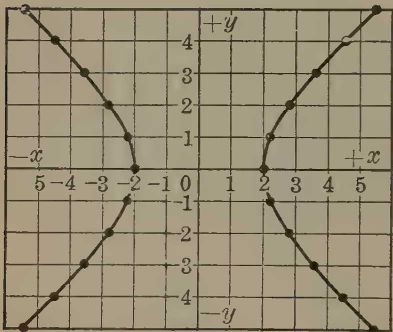


7. Plot the graph of $2x^2 + 3y^2 = 72$. First solve for x or y. Make the table of values as usual. The curve is an *ellipse*.

8. Plot the graph of $9x^2 + 4y^2 = 36$.

9. Plot the graph of $x^2 - y^2 = 4$. First solve for x or y, then make a table of values. The curve is a *hyperbola*. Notice

that it has two axes of symmetry as do the circle and the ellipse, while the parabola has but one. Discuss the possibility of drawing one of the branches of a hyperbola only and so being misled into regarding it as a parabola. Explore the field between $x = 2$ and $x = -2$. How do the values of y within these limits indicate that the graph does not exist there?

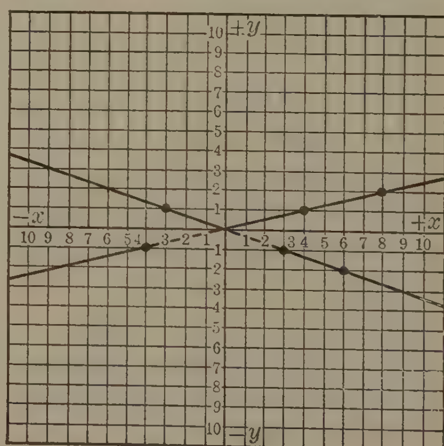


10. Plot the graph of $2x^2 - y^2 = 10$.

11. Plot the graph of $xy = 1$.

12. Plot the graph of $x^2 - xy + 12y^2 = 5$.

13. Plot the graph of $x^2 - xy - 12y^2 = 0$. Note that the entire expression in x and y is factorable and the equation may be written $(x-4y)(x+3y)=0$. It follows that either $x-4y=0$ or $x+3y=0$. Plot the graphs of both of these equations. Verify the graph by plotting coördinates for the original equation in the usual manner. Contrast with the preceding example.



14. Plot the graph of $x^2 + 6y^2 = 7xy$.

*15. **The form of a quadratic equation and the form of its**

graph. The general form of a quadratic equation in two variables is $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$, and an illustration is $2x^2 - 5xy + 8y^2 - 9x + 3y + 4 = 0$. When one or more of the constants is zero, the equation takes on some special form as illustrated in examples 1–12 above. It is sometimes both easy and convenient to predict the form of the graph of such an equation.

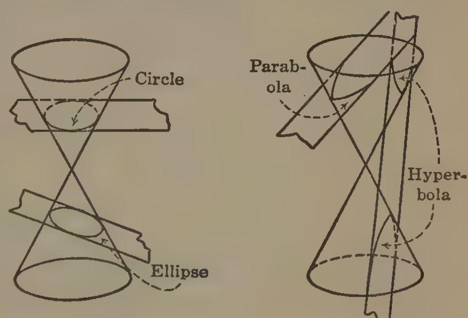
$$\left.\begin{array}{l} x = ay^2 \pm by \pm c \\ x = ay^2 \pm b \\ x = ay^2 \end{array}\right\} \text{The graphs are parabolas.}$$

$$\left.\begin{array}{r} ax^2 - by^2 = c \\ xy = a \end{array}\right\}$$ The graphs are hyperbolas (if a and b have the same sign).

$ax^2 + by^2 = c$ The graph of this equation is an ellipse (if a and b have the same sign) with its center at the origin. Show how its intercepts may be found.

$ax^2 + ay^2 = c$ The graph of this equation is a circle with its center at the origin, and radius $\sqrt{\frac{c}{a}}$.

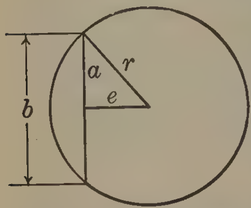
16. Conic sections. The curves mentioned in the preceding examples are called conic sections because they may be developed by passing planes through cones as illustrated here. Portions of ellipses are often seen in arches of bridges or other architectural structures. The path of a comet around the sun is a parabola or a hyperbola. Find other applications of the conic sections.



Try Exercise 53, F, page 452.

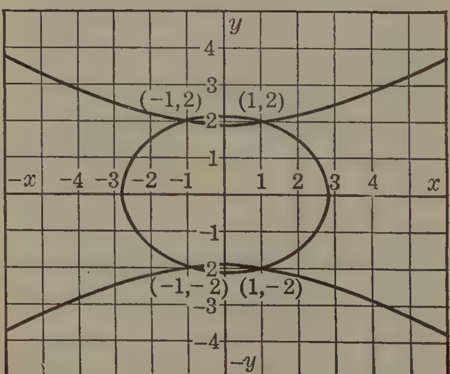
***17. A study of a geometric illustration of quadratic dependence.** In the accompanying figure, since $b = 2a$, it follows that

$b = 2\sqrt{r^2 - e^2}$. Explain. State this fact verbally thus: In any circle the length of a chord is equal to. . . If r is a constant, increasing e has what effect upon b? Increasing b has what effect on e? Implied in this algebraic formula are the following two geometric theorems: *In the same circle, or in equal circles, if two chords are equal, they are equally distant from the center. If two chords are unequal, the one more remote from the center is the smaller*, and their converses. Which is the more compact form of the statement, the algebraic or the verbal? Which gives the more information? Let r be constant, as $r = 10$, and make a table of values for b and e. Let $b = 6, 8, 10$, and 12. Solve the original formula for e; make an equivalent verbal statement; check your work.



18. Graphic solution of pairs of equations involving quadratics. Contrast the graphic solution of a single equation, as in example 2, page 256, and the graphic solution of a pair of equations in two unknowns. Can the graphs of two equations intersect in more than one point? Solve graphically the following pair of equations: (1) $2x^2 - y^2 = 14$ and (2) $x - y = 1$.

Plan of solution: Make a table and plot a graph for each equation. Find the roots at the intersections of the graphs. Set down the roots *in pairs* and check them by substitution. The roots are $x = 3$, $y = 2$; and $x = -5$, $y = -6$.



19. Solve graphically ① $4x^2 + 7y^2 = 32$ and ② $11y^2 - 3x^2 = 41$.

20. Solve graphically and estimate the roots to the nearest tenth ① $x^2 + y^2 = 25$ and ② $2x - 3y = 5$.

***21.** What point is common to the graphs of

$$\begin{aligned} &① \quad x^2 + y^2 = 169 \\ &② \quad xy = 60 \\ \text{and } &③ \quad x + y = 17? \end{aligned}$$

Try Exercise 53, G, page 453.

Other Types of Dependence

***1. Trigonometric dependence.** Plot the graph of $y = \sin x$.

Plan of attack: Complete the table and plot as usual. Find the values of y in the table of sines of angles. The sine of any angle between 90° and 180° is equal to the sine of its supplement. The sine curve is a typical wave. It is used in studies based upon wave motion, such as studies of sound, light, and electricity.

x	y (or $\sin x$)
0°	0
10°	.17
20°	.34
30°	
40°	
60°	
80°	
90°	
100°	
120°	
150°	
180°	

***2.** Plot the graph of $y = \cos x$. (The cosine of any angle between 90° and 180° is minus the cosine of its supplement.)

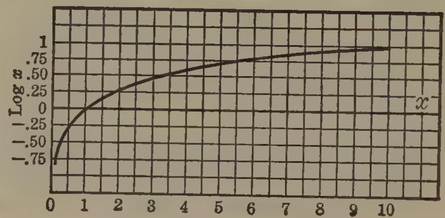
***3.** Plot the graph of $y = \tan x$. (The tangent of any angle between 90° and 180° is minus the tangent of its supplement.)

***4. Logarithmic dependence.** Plot the graph of $y = \log_{10} x$. See page 130.

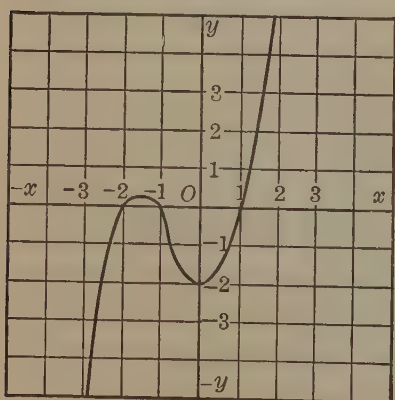
Plan of work:

x	y, or $\log_{10} x$
0.1	-1
0.2	-0.7
0.3	-0.5
0.4	-0.4
0.5	-0.3
0.6	-0.2
1.0	0
2.0	0.3
3.0	0.47
4.0	0.6
5.0	
6.0	
10.0	1.0
30.0	1.4
40.0	
100.0	2.0

Explain the table and complete it. Show how the graph was constructed from the table.



What is the logarithm of 1? What is the sign of the logarithm of any number numerically less than 1? Notice that the graph can come as near to the x-axis as we choose to make it except that it can never touch the x-axis. When we interpolate in a table of logarithms, we assume that changes in the logarithm are directly proportional to changes in the number. Is this exactly true? Which part of the graph is approximately straight? Explain why interpolation does not give accurate results if the tabular differences are very large.



***5. Cubic dependence.** The equation $(x-1)(x+1)(x+2)=0$; that is, $x^3+2x^2-x-2=0$, has, evidently, the roots -1, 1, and -2. This means that the graph

must cross the x-axis at what three points? Make a table of values. Explain the accompanying graph of $x^3 + 2x^2 - x - 2 = y$.

*6. Plot the graph of $x^3 - 2x = y$.

*7. Plot the graph of $x^3 - 2x - 4 = 0$.

8. Review example 10, page 453.

Part II. Statistics and Statistical Graphs

The word *statistics* meant originally a collection of facts relating to a state (nation) and its inhabitants. This meaning has gradually broadened to include almost any collection of facts which can be expressed in numbers, and now we see statistics of many kinds displayed in a great variety of ways for a great variety of purposes. Of our four methods of representing relationship, the verbal, the tabular, and the graphic are commonly used in displaying statistics, and formulas are made use of whenever the relationships involved are governed by known mathematical laws.

The study of statistics has led to one of the great modern advances in applied mathematics. Parts of the theory of statistics are based upon the very interesting mathematical laws of probability. These laws are not at present included in a second course in algebra, and we must confine ourselves here to a few of the elementary facts about statistics with which every one should be acquainted.

Terms Used in Elementary Statistics

1. Consider the numbers 418, 156, 156, 140, 118, 115, 115, 115, 118, 156, 126, 125, 156, 156, 156. Find their *average*; that is, $\frac{1}{n}$ of the sum of the n numbers. Arrange the numbers in a *frequency table*; that is, in a table which shows that 156 occurs

six times and so on down to the numbers which occur but once. Find the *mode* or the fashionable number; that is, the number which occurs most frequently. Arrange the fifteen numbers in *order of size*, the largest first and the smallest last. Find the *median*; that is, the middle number when they are arrayed in order of size. (If there is no middle number, take the average of the one above and the one below the middle.) Find the *lower quartile*; that is, the number midway between the smallest number and the median. Find the *upper quartile*; that is, the number midway between the median and the largest number. Explain the significance of the word *quartile*. Notice that the average, the mode, and the median each give some information concerning the nature of the group of numbers.

2. A collection was made in a class to raise money for flowers. The teacher gave $1.00; one pupil gave $0.50; and twenty other members of the class gave 10 cents each. An absentee who also wanted to give asked what the average gift was. What was it? What he really wanted to know was the amount of the modal gift. What was it?

3. A jury was to decide the amount of damages to be paid by A to B who had been injured by A's automobile. Each juror wrote on a slip of paper the amount which he considered just. These amounts were: $1000, $100, $1500, $1000, $1200, $250,000, $1200, $1100, $1100, $100, $800, and $1000. Find the average, the mode, and the median. Which amount do you think best expresses the consensus of opinion of the jury? Why?

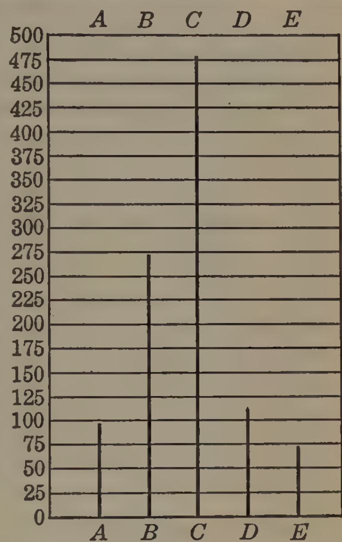
Statistical Graphs

The bar graph and the broken-line graph.

1. In a large high school, marks were given in senior English as follows: 96 A's, 274 B's, 476 C's, 112 D's, and 72 E's. Represent these numbers by bars.

Discussion: Tell how the largest number in this list was used

in determining the vertical scale. Tell how the horizontal scale was selected. How would cross section paper help in the construction of the graph?



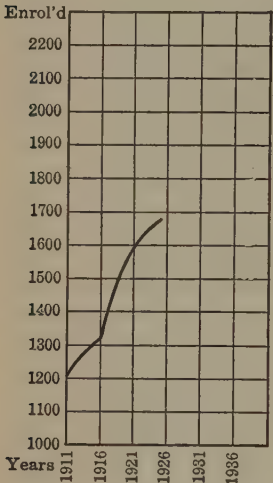
2. In senior mathematics there were in October 50 A's, in November 72, in December 70, in January 60, in February 60, and in March 76. Represent these numbers by a bar graph. (Narrow bars are easier to construct than wide bars. If broad bars are used, it is a good plan to make the bars as broad as the spaces between them.) Label the graph neatly and sufficiently. Justify the scales you have used.

3. Show how the values of A in the formula $A = (n - 2)180$ in which n is a positive integer may be represented by a bar graph. Construct the graph from $n = 3$ to $n = 7$.

4. Reconstruct the graph of Exercise 2 in the following manner: Locate the top point of each bar, do not draw the bars, but instead connect these points in succession by a broken line. Notice that the bar graph contrasted the numbers in the group; the broken line graph does more; it suggests the *change* from one month to the next. In this illustration it suggests that the number of pupils in the A group was gradually increasing between the October and the November marks, etc. Explain. In which of Exercises 1–3 would a broken line graph be appropriate?

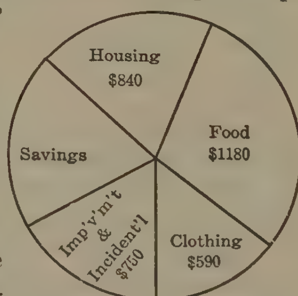
5. A high school opened with an enrollment of 1200 pupils.

At the end of five years the enrollment was 1375. Five years later it was 1600, and five years after that 1700. Represent these numbers by a curved-line graph. Make the construction as in Exercise 3, above, but draw a curve instead of a broken line through the points. Estimate the approximate probable attendance at the end of another five years. Can this number be determined with "mathematical certainty"?



***6. The circle graph.** A family's experience with a budget showed that the income of $4200 was expended as follows: housing, $840; food, $1180; clothing, $590; improvement and incidentals, $750; savings, $? Supply the missing numbers in the table and explain the construction of the graph. What are the advantages of the circle graph in representing a whole divided into parts?

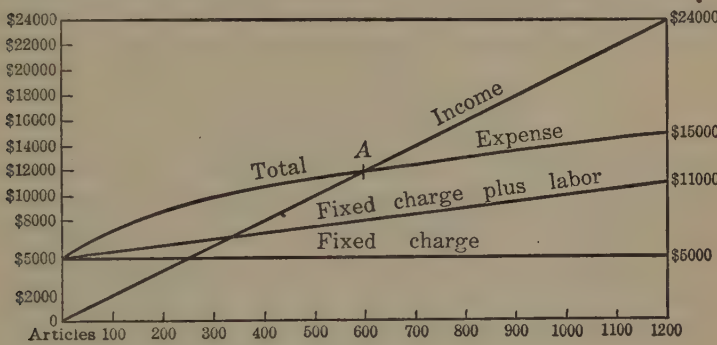
Classification	$	% (to nearest per cent)	Degrees (to nearest degree)
Housing			
Food			
Total	4200	100	360



*7. Contrast the uses of the three kinds of graphs called for in the preceding examples. Notice also that a rectangle may be divided into smaller rectangles so as to serve the same purpose as the circle graph. See page 186.

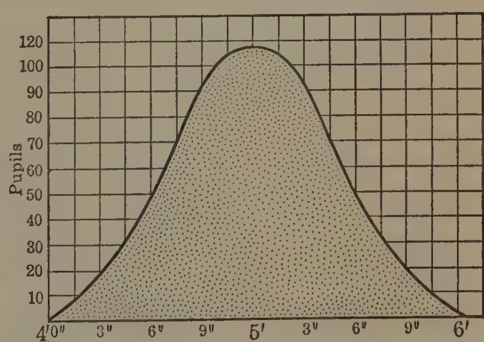
*8. A manufacturer has to pay $5000 a month fixed charges; $5 for labor for each article manufactured, $2000 for operating

expenses if he makes 100 articles a month, $3000 if he makes 200; $3500 if he makes 400, and $4000 if he makes 1200 a month, which is his maximum. If he sells each article for $20, how many must he sell per month to avoid loss? Explain in detail the following graphic solution. Show how the point A indicates the answer. What portions of the diagram represent possible loss? possible profit?



Try Exercise 53, H, page 454.

***9. The curve of normal distribution.** Suppose that the pupils of the tenth grade in a large city school were arranged in squares according to their heights as indicated in the figure, in which each dot represents one child. The distribution would be somewhat as shown, with



many in the middle and few at the ends. Many other natural

characteristics show the same kind of grouping: consider, for instance, the weight of each grain in a load of ungraded wheat; the length of each ear of corn in a ten-acre field; the ability of the pupils in a large school system; etc. The curve, then, which is roughly outlined by the dots is of importance in the study of statistics. It is called the *curve of normal distribution*. Its equation is $y = e^{-x^2}$, in which $e = 2.7$ approximately.†

Plot the graph of this curve between the values $x = 2$ and $x = -2$. (Let $x = -2, -\frac{3}{2}, -1, -\frac{1}{2}, 0$, etc.)

Reviews and Tests

Test A. Test on Chapter VIII

***1.** Make an outline showing the headings of this chapter, and tell briefly what you have learned under each heading.

2. What is meant by *a pair of inconsistent equations in two unknowns*? How are they shown graphically? What are *dependent equations*? How are they shown graphically?

***3.** By Boyle's Law the volume of a gas at constant temperature varies inversely as the pressure. 1060 cubic inches of gas under 14.7 pounds pressure will occupy what volume when the pressure is increased to 15.3 pounds?

4. Study the following table and complete the statements:

$a =$	1	2	3	5
$b =$	16	64	144	400

If $a = 7$, $b = ?$ $b = 1296$, $a = ?$ $\frac{a^?}{a_1^?} = \frac{b}{b_1}$

If $b = ka^x$, $k = ?$ and $x = ?$

Check the answers by means of the k formula.

b varies as ? a varies as ?

† The number which mathematicians represent by the letter e is the sum of a certain infinite series. It is an irrational number. It is used as the base in what is called the "natural system" of logarithms and has other uses in higher mathematics.

***5.** Solve graphically $x^3 - 4x = 0$. Solve graphically on the same axes $x^3 - 4x = y$ and $2x + y = 21$.

***6.** A one-celled animal absorbs its food through its surface. Show that if it continues to grow, it will eventually be unable to absorb sufficient food. The animal overcomes this difficulty by separating into two cells. Facts such as these show how mathematical laws determine the make-up of the universe.

Test B. Problem Test

Review the "Plan of Problem Solution," page 169.

***1.** If an animal's strength is proportional to the square of his length, and an animal 4 ft. long can pull 1250 pounds, how much can an animal 4.38 ft. long pull? Answer to three-figure accuracy.

2. The intensity of illumination from a source of light varies inversely as the square of the distance. A book is 4 ft. from the light; how far from the light must it be placed in order to receive double the illumination?

3. The area of a circle varies as the square of its diameter. If the area of a circle whose diameter is 4″ is $\frac{88}{7}$ sq. in., what is the diameter of a circle with an area of $\frac{77}{2}$ sq. in.?

4. The distance that a body falls from rest varies as the square of the time elapsed. How far will a body fall in 8 seconds if it falls 256 feet in four seconds?

***5.** In each problem 1–4 express the variation in the k form, and find the value of k. Tell why mathematicians make use of the k form for expressing such relationship.

***6.** The force, F, necessary to lift a weight, W, by means of a certain machine is given by the formula $F = s + tW$, in which s and t are constants depending on the friction in the machine. (*a*) Is the variation linear or quadratic? (*b*) Is it proportional

variation? (*c*) If a force of 28 pounds will raise a weight of 5 pounds and a force of 38 pounds will raise a weight of 7 pounds, find s and t. (*d*) What force will raise a weight of 18 pounds?

7. A beam supported at each end and carrying a given load at the center sags in proportion to the cube of its length. A beam eight feet long and so loaded sags $1\frac{1}{4}$ inches. (*a*) How much will a beam six feet long sag under the same load? (*b*) Twenty feet long?

8. A grocer had on hand thirty pounds of tea which had been selling at 80 cents a pound. The price was unpopular, so he decided to reduce it to 64 cents a pound by adding tea that had been selling at 40 cents a pound. How much 40-cent tea should he add?

***9.** If I put $12,000 in a savings bank that pays $4\frac{1}{4}\%$ compounded semi-annually, how much will it amount to in two years?

10. Two and one half hours after a detachment of engineers were sent out from camp in a motor truck that averages 12 miles an hour, it became necessary to change their destination. How fast would a messenger have to travel on a motorcycle to overtake the truck by the time it reached a crossroads 48 miles from camp?

Test C. Review of Certain Fundamental Ideas

Read paragraph 5 of *How to Study Algebra*, page xx.

1. Show that a signed number means both a distance and one of two opposite directions. What is meant by the absolute value of a number?

2. What is the value of $\sqrt{x^{-3}}$ when $x = 4$? Write x^{-a} without a negative exponent and show that this transformation is in accordance with the laws of multiplication and of division of exponents. Find the value of $5^0 - 27^{\frac{1}{3}} + \sqrt{3} - 12^{\frac{1}{2}}$.

3. How is any term of an arithmetic progression derived from the one immediately preceding it? Answer the same question for a geometric progression. Find three numbers in arithmetic progression such that their sum is 6, and when 4 is added to each of the first two and 12 to the third, the new numbers are in geometric progression.

4. What kind of series may have a "sum to infinity"? Under what conditions? Derive the formula for finding such a sum. Find the value of 2.183183...

5. Factor: (*a*) $x(a-b)-y(b-a)$
(*b*) $2-7x+3x^2$

6. Perform the operations indicated:

$$\left[\frac{2a}{b+2a}-\frac{b}{2a-b}+\frac{3a^2}{b^2-4a^2}\right]\div\frac{4a-b}{(2a-b)^2}$$

7. Solve and check: $x-y=a-b,\quad ax-by=2a^2-2b^2$

***8.** Solve and check:

$$\tfrac{1}{2}(a-2b)-\frac{3a}{4}-\frac{4b}{3}+6\tfrac{1}{4}=0$$

$$5a+5b-\tfrac{1}{3}(b-2a)=\frac{\tfrac{2}{5}(-4a+b)}{2}$$

***9.** Solve and check:

$$\frac{x+y+1}{x-y-1}=a \qquad \frac{x+y+1}{x+y-1}=b$$

10. Solve by formula: $x^2-(e-2f)x-2ef=0$

11. A regular five-sided polygon has a perimeter of 152 inches. Find its area. (*Hint:* From the center of the pentagon draw radii to two adjacent vertices and consider the isosceles triangle thus formed.)

12. A beam supported at both ends is required to carry a load of 5000 pounds at its center, and must be 24 ft. long and

6 in. wide. When a beam 4 in. deep is used, it sags $2\frac{1}{2}$ inches. What depth must be used if the beam is not to sag more than one inch? (Assume that the sag varies inversely as the square of the depth.)

***13.** A fast express train runs at 48 miles an hour from A to B, a distance of 216 miles. It goes by way of C, a city 176 miles from A, at which it does not stop. Thirty-five minutes before it passes C, however, a local train leaves C to provide at B a connection with the express and to reach B ten minutes before the express. At what speed must the local train be scheduled in order to make this connection?

***14.** A theater balcony seats 250 persons in 10 rows of seats, each row containing two fewer seats than the one in front of it. If seats in the first five rows sell at 50 cents each, and the rest at 25 cents each, what is the maximum income from the balcony?

15. The minimum speed needed to keep in the air an airplane of given design varies as the square root of its length. If the minimum speed for a 24 ft. airplane is 40 miles per hour, what is the minimum for a similar 36 ft. plane?

16. In the formula $y = \dfrac{x^2 - 2ax - 12}{a - 2}$ let $a = 4$ and plot the graph of the resulting equation. From the graph read the value of x when $y = 10$. For what values of x is y greater than zero? equal to zero? less than zero?

CHAPTER IX

FURTHER STUDY OF SYSTEMS OF LINEAR EQUATIONS

Introductory Questions

1. What is the derivation and meaning of the word *eliminate*?

2. What is meant by *elimination by combination*? *Elimination by substitution*?

3. What is meant by *solving* a pair of simultaneous equations in two unknowns?

4. Describe the graphic solution of a pair of linear equations in two unknowns. (Use the word *locus* and its plural *loci*.)

5. Name conditions under which a pair of equations in two unknowns have no solution.

6. How would you expect to solve a system of three equations in three unknowns?

Elimination by Substitution

1. Solve the equations $x = \dfrac{8 + 3y}{4}$ and $7x - 4y = 19$ by substituting the value of x from the first equation in the second.

Solve by substitution:

2. ① $11x + 10y = -35$
② $x - 4y = -67$

Plan of solution: In one equation express one unknown in terms of the other. (Select the equation and the unknown so as to make the work as easy as possible.) Substitute the value thus found in the other equation.

③ $x = 4y - 67$ ② $+ 4y$

④ $11(4y - 67) + 10y = -35.$ ③ in ①.

Complete and check.

3. $4a - 5b = 31$
$3a + 2b = 6$

4. $7a + 10a' = -21$
$4a - 6a' = 70$

5. $4a - 15b = 7$
$a = \dfrac{3 + 9b}{2}$

6. $7x - 177 = 3y$
$4x - 11y = -1$

7. $3a - 4b = 6$
$\frac{1}{2}a + \frac{2}{3}b = 2$

8. $4w - 21z = 16$
$16w + 15z = 18$

If more practice is needed, solve by substitution some of the examples of Exercise 9, page 355.

9. When systems are solvable. An investigation. Study the solutions suggested below and the discussion of them, and then sum up your conclusions in your own words.

(*a*) $x + y = 10$ (*b*) $2x + y = 10$
(*c*) $\frac{2}{3}x + \frac{2}{3}y = 20$ (*d*) $4x + 4y = 40$

Explain the result of attempting to solve as a pair of linear equations (*a*) and (*b*); (*a*) and (*c*); (*a*) and (*d*); (*b*) and (*c*). Which of these pairs of equations are *inconsistent*? Can a system of inconsistent equations be solved? How is the condition of inconsistency illustrated in a graph? Which of these pairs of equations are *dependent*? In a pair of dependent equations is the information sufficient to permit a unique solution? How is the condition of dependency illustrated in a graph? Whenever a system of equations has an unlimited number of solutions, it is said to be *indeterminate*. In general, one equation in two unknowns, and one or two equations in three unknowns are indeterminate. Linear systems are determinate when the number of independent equations is the same as the number of unknowns and when no two of the equations are inconsistent.

*When two equations are in the form

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

it is usually easy to discover whether they are dependent or inconsistent by inspecting the ratios of the corresponding constants. Consider also the following facts: If we eliminate y, and write

$$(a_1b_2 - b_1a_2)x = c_1b_2 - b_1c_2$$

it is evident that if $a_1b_2 - b_1a_2 \neq 0$, there is one solution for the pair of equations; if $a_1b_2 - b_1a_2 = 0$ and $c_1b_2 - b_1c_2 \neq 0$, there are no solutions; and if $a_1b_2 - b_1a_2 = 0$ and $c_1b_2 - b_1c_2 = 0$, there is an indefinite number of solutions. In this connection, see Examples 1–4, page 254.

Systems of Literal Equations

Solve for x and y and check your answers:

1. $ax + by = c$ Show that your answers may be used as formulas
$dx + ey = f$ for the solution of linear pairs. Solve by means of these formulas Exercise 2 on page 273.

2. $\dfrac{x}{r} + \dfrac{y}{s} = 1$

$sx - ry = 0$

3. $\dfrac{3\,ax - 2\,by}{5\,b} = a$

$\dfrac{2\,x + by}{3 + a} = 2\,b$

Try Exercise 55, page 459.

Systems in more than two unknowns

1. Solve and check:

① $4\,x - 5\,y + 8\,z = 18$ ② $2\,x + 3\,y + z = 4$ ③ $x + 2\,y + 2\,z = 7$.

Plan of solution: Eliminate x in ① and ②; eliminate *the same unknown* in ② and ③; solve the resulting pair for y and z.

④ $4\,x + 6\,y + 2\,z = 8$ ② $\times$ 2

$$\begin{array}{llr} ⑤ & \qquad -11y+6z=10 & ①-④ \\ ⑥ & 2x+4y+4z=14 & ③\times 2 \\ ⑦ & \qquad\quad -y-3z=-10 & ②-⑥ \end{array}$$

Complete. Check by substitution of the roots in all three original equations. Discuss the importance of a systematic arrangement of work in such solutions as this one. Explain the importance of the italicized words in the "plan of solution."

2. Solve and check:

$$\frac{5x+3y}{2}=z+0.5 \quad 3x+4z=14 \quad \text{and} \quad \frac{2(y-1)}{7}=x-6$$

***Try Exercise 56,** page 461.

Problem Solution

Recall the systematic method of attack in problem solution which was developed in Chapter V. Solve and check each of the following problems, using one, two, or three unknowns as you think best.

1. Try to solve problem (*a*) alone; problem (*b*) alone; (*a*) and (*b*) together as a single problem; (*a*), (*b*) and (*c*) together as a single problem. Discuss the results.

(*a*) Five boys and 10 men were paid \$55 for a day's work. How much did one boy receive? One man?

(*b*) At the same rates, 2 boys and 5 men were paid \$26.50 for a day's work. How much did one boy receive? One man?

(*c*) At the same rates, 4 boys and 2 men received \$28 for a day's work. How much did one boy receive? One man?

2. Three cities A, B, and C are so located that the distance from A to C by way of B is 50 miles; the distance from A to B by way of C is 70 miles; and the distance from B to C by way of A is 60 miles. How far apart are the cities? Make a drawing.

3. \$5.55 was paid in dimes, quarters, and halves, 18 coins in all. The number of halves was equal to the number of dimes and quarters together. How many coins of each were used?

4. Can \$2.65 be paid with a total of 26 coins, nickels, dimes, and quarters, if there must be 4 more dimes than nickels and quarters together?

5. An office boy in buying nine dollars' worth of stamps wants 100 more one-cent stamps than threes and fives together and twice as many fives as threes. How many of each kind must he buy?

6. A contractor employed 3 plumbers for 5 days and 2 helpers for 6 days and paid them a total of \$201. He employed 5 plumbers for 6 days and 4 helpers for 8 days and paid them a total of \$450. What were the daily wages of each?

7. Can you find the prices of corn, oats, and rye from the following information? A bill for 5, 6, and 8 bu. respectively amounted to \$10.30; another bill for 3, 5, and 8 bu. respectively to \$8.75; two bu. of oats and rye mixed half and half sold for as much as $1\frac{2}{3}$ bu. of corn.

***8.** A, B, and C subscribed \$100 together. If C had put in \$2 more than he did, and B $\frac{1}{10}$ more than he did, A could have completed the sum by subscribing $\frac{1}{10}$ less. If C had put in \$18.50 more and B $\frac{1}{8}$ more than he did, A could have put in $\frac{1}{8}$ less. How much did each subscribe?

***9.** Is the following information sufficient to determine the time it would take A, B, and C each working alone to do a certain job? A and B can do it together in 12 days. After they get it three fourths done, however, they call on C to help them and thus save one day. C can do as much work in 5 days as A can in 6.

10. A quantity of milk sufficient to fill three cans of different sizes will fill the smallest can five times, the largest can twice with four gallons to spare, or the second twice with 8 gallons to spare. What is the capacity of each can?

11. The sum of a certain number and its reciprocal is $\frac{13}{6}$. Find the number.

12. The sum of the reciprocals of two consecutive odd numbers is $\frac{12}{35}$. Find the numbers.

***13.** If seven buns, three sandwiches, and one bottle of pop cost 14 cents, and ten buns, four sandwiches, and one bottle of pop cost 17 cents, can you find the cost of one bun, one sandwich, and one bottle of pop?

***14.** The following problem arose in court, and the judge required that the accountants solve it.† Two firms, A and B, failed at the same time, and each owed the other. A owed B \$12,500 and was to pay the same part of this debt as of his other debts. B owed A \$5200 and was to pay the same part of this debt as of his other debts. The questions were: (1) What per cent of his debts could A pay? B? (Answer to the nearest hundredth of a per cent.) (2) How much had A to pay to B and B to pay to A? (Answer to the nearest dollar.)

	Assets	Liabilities
A	\$200,000 + what he received from B	\$300,000
B	\$90,000 + what he received from A	\$250,000

Plan of solution: Let x represent the per cent A paid and y the per cent B paid. The other numbers involved are clearly suggested in the problem. One equation may be based on the relation \$200,000 + \$ received from B = $x\%$ of A's liabilities. Explain.

† I am indebted for this problem to W. S. Schlauch, of the High School of Commerce, New York City.

Reviews and Tests

Test A. From College Examination Papers

1. Two machinists working at similar machines are paid in proportion to the number of parts turned out. A turns out 24 parts in the same time that B turns out 32 parts. A earns \$36 a week. How much does B earn?

2. In the formula $r_2 = \dfrac{-b - \sqrt{b^2 - 4\,ac}}{2\,a}$ $a = 10$, $b = -3$, $c = -11$. $r_2 = ?$ Extract the square root to the nearest tenth.

3. A boy skated across a lake with the wind at the rate of 12 miles an hour. Returning, against the wind, he went at the rate of only 4 miles an hour. He made the entire trip in $1\frac{1}{2}$ hours. How wide is the lake?

4. State whether each of the following statements is true or false.

(*a*) If x is an integer, $x^2 - 4\,x + 4$ is always greater than $+1$.

(*b*) $\dfrac{x^2 - 2\,xy + y^2}{x + y} \cdot \dfrac{2(x + y)}{2(x - y)(x - y)} = 0$

(*c*) $16 + 24\,x + 9\,x^2$ is a perfect square.

(*d*) If $\dfrac{1}{N} = \dfrac{1}{C} + \dfrac{1}{R}$, then $N = \dfrac{CR}{C + R}$.

(*e*) The expression $4\,a$ means that a is used 4 times as a factor.

Test B

1. Solve the following pair of equations by two methods of elimination. Solve them graphically:

$$2\,x - 3\,y = -7 \qquad 4\,x - 5\,y = -9$$

2. From the formulas $s = 2\,rh$ and $h = \frac{1}{2}\,s\sqrt{3}$, find r.

*3. In the formulas $s = \frac{n}{2}(a + l)$ and $l = a + (n - 1)d$, eliminate a and solve the resulting equation for l.

4. What term must be added to $x^2 - 6x$ to make the expression a perfect trinomial square? Answer the same question for $4x^2 - 20x$. For $\frac{25x^2}{4} - \frac{10x}{3}$.

5. Simplify $\pm\sqrt{\frac{3}{2}}$ and $\sqrt{\frac{-4ac + b^2}{4a^2}}$. Unite $-\frac{c}{a} + \frac{b^2}{4a^2}$.

6. If $r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ find $r_1 + r_2$ and $r_1 r_2$.

7. Solve $3\sqrt{x} + 4\sqrt{y} = 18$ $\quad \sqrt{x} - 5\sqrt{y} = -13$.

8. What is the value of $\sqrt[3]{-8}$? $\sqrt[3]{-27}$? -4^2? $-(-2)^2$?

If $x = -64$, what is the value of $\frac{1}{16}\sqrt[3]{-x^2}$?

9. Find by logarithms the value of $\sqrt{\frac{2.365}{\cos 17.65°}}$.

10. Given the formulas $V_1 = V_0 - at$, and $s = V_0 t - \frac{1}{2}at^2$, solve the first for t, substitute in the second the value of t as found in the first, and solve the new equation for V_0.

11. Square both sides of the equation $\sqrt{4x + 5} = 3x - 10$ and solve for x. Check both roots of the resulting equation. Do both satisfy the original equation? (Remember the meaning of $\sqrt{}$.) Explain.

CHAPTER X

FURTHER STUDY OF QUADRATIC EQUATIONS

Part I. Solution

Introduction

This chapter gives you the opportunity to try your hand at two very interesting and important kinds of tasks: 1, the summing up in a systematic way of what you know about a topic, and 2, the drawing of general conclusions about it. Ability to do these things is one mark of a trained mind.

1. Define a quadratic equation in one unknown and give the derivation of the word quadratic.

2. How many roots has a quadratic equation in one unknown? What kinds of numbers may these roots be? (See pages 40 and 256.)

3. Is it true that changes in the constants in a quadratic equation result in changes in the roots of the equation? Illustrate.

Questions to have in mind as you study this chapter

4. How does the chapter illustrate the scientific method of study?

5. How does the chapter show the help which algebraic symbols give us when we are thinking about numbers?

6. In Chapter VI was emphasized the importance of the ability to recognize the form of an algebraic expression. Where may this ability be used in Chapter X?

Derivation of the Quadratic Formula

1. The task is to solve the general quadratic equation $ax^2 + bx + c = 0$ and thus show that its roots are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Factoring by inspection will not accomplish this result, therefore we factor by *completing the square.* What are the criteria of a perfect trinomial square? What must be added to $x^2 - 10x$ in order to form a perfect square? To $x^2 + 5x$? To $4x^2 - 12x$?

2. Completing the square. Solve by the method of completing the square ① $x^2 - 8x - 20 = 0$

Plan of solution:

② $x^2 - 8x = 20$ ① + 20

③ $x^2 - 8x + 16 = 36$ ② + 16. Since the 8 is a double and the 16 the square of the same number, how may the 16 be determined when the 8 is known?

④ $x - 4 = \pm 6$ $\sqrt{③}$ Notice that *both square roots* of 36 must be taken in order to obtain both roots of the equation.

Complete and check.

3. Solve by completing the square ① $3x^2 - 13x - 10 = 0$.

Plan of solution:

② $x^2 - \frac{13}{3}x = \frac{10}{3}$

③ $x^2 - \frac{13}{3}x + \frac{169}{36} = \frac{10}{3} + \frac{169}{36}$. Show that $\frac{139}{36} = (\frac{1}{2} \times \frac{13}{3})^2$.

Show that the left-hand member is a perfect square.

④ $x - \frac{13}{6} = \pm \frac{17}{6}$ $\sqrt{③}$

Complete and check. Discuss the solution in case the roots are irrational; in case they are imaginary. Illustrate.

4. Solve by completing the square not less than five examples from Exercise 14, page 360.

5. The quadratic formula. Solve by completing the square:

① $ax^2 + bx + c = 0.$

② $x^2 + \dfrac{bx}{a} = -\dfrac{c}{a}$ ① $\div a$

③ $x^2 + \dfrac{bx}{a} + \dfrac{b^2}{4a^2} = -\dfrac{c}{a} + \dfrac{b^2}{4a^2} \equiv \dfrac{-4ac + b^2}{4a^2}$ State and justify a general rule for finding the third term of the left-hand member.

④ $x + \dfrac{b}{2a} = \pm\sqrt{\dfrac{-4ac + b^2}{4a^2}} \equiv \pm\dfrac{1}{2a}\sqrt{b^2 - 4ac}$

⑤ $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Substitute these roots in the original equation and thus check your work.

6. Solve by completing the square: $px^2 + qx + s = 0$, and check by substituting the roots in the original equation.

Four Methods of Solving a Quadratic Equation

1. Solve $4x^2 + 4x - 15 = 0$, (*a*) graphically, (*b*) by factoring by inspection, (*c*) by completing the square, and (*d*) by formula.

2. Discuss the relative merits of the four methods of solving quadratics. Notice that the graphic method is instructive but not always efficient. Much of the value of graphs lies in the added insight which they give.

3. Solve and check, $nx^2 + sx + t = 0$, using the method of completing the square.

Special Quadratic Equations

1. Zero constants. Incomplete quadratics. Consider the equations:

$$\begin{aligned} 4x^2 - 48x - 81 &= 0 \\ -48x - 81 &= 0 \\ 4x^2 \qquad\quad - 81 &= 0 \\ 4x^2 - 48x \qquad\quad &= 0 \end{aligned}$$

Each is in the form $ax^2 + bx + c = 0$, but in the second $a = 0$, in the third $b = 0$, and in the fourth $c = 0$. Solve each equation.

The first equation has two roots. It is a complete quadratic equation in standard form.

The second is not a quadratic at all. Why? How many roots has it?

The third may be solved by factoring as the difference of two squares, or by writing $4x^2 = 81$ and extracting the square root of both members. In using the latter method both square roots of 81 must be taken in order to obtain both roots of the equation. Explain. A quadratic equation in which $b = 0$ is sometimes called an *incomplete quadratic equation.*

The fourth equation may be factored. The roots are 0 and 12. Notice that if you divide the equation by $4x$ you may fail to discover the zero root. State two cautions based upon your study of the third and fourth equations above.

2. Equivalent equations. Extraneous roots. Two equations are said to be equivalent if all the roots of either are roots of the other. In general when we correctly transform any equation as in solving it, we obtain a series of equivalent equations, and all the roots of the final equation are all the roots of the original equation. However, there are exceptions which must be considered. Study the transformations below in order to learn how an *extraneous* root can be introduced, that is, a root of one of the transformed equations which is not a root of the original equation. Check the roots obtained. Which root is *extraneous*? Of which of the equations is it a root? By what process was it introduced?

① $x + 1 = -1$

② $x^2 - 2x - 3 = -x + 3$ ① $\times\ x - 3$

③ $x^2 - x - 6 = 0$

④ $x = 3$ or -2 Are these both roots of the original equation?

In the same way study the following transformations:

3. ① $\sqrt{x+7} = 5 - x$

② $x + 7 = 25 - 10\,x + x^2$ ① squared

③ $0 = x^2 - 11\,x + 18$

④ $x = 2$ or 9. Check both roots.

4. Let ① $x = 1$

② $x^2 - 2\,x = x - 2$ ① $\times\, x - 2$

③ $x^2 - x = 2\,x - 2$ ② $+\, x$

④ $x(x - 1) = 2(x - 1)$

⑤ $x = 2$ ④ $\div\, x - 1$

⑥ $1 = 2$ ① sub. in ⑤

Point out the error.

5. Solve $x^3 + 3\,x^2 - 10\,x = 0$. (First divide by x. Notice that there are three roots.)

6. Tell how extraneous roots may be introduced by transforming equations.

Cautions. Supply the missing words and explain the following cautions. Keep them in mind when transforming equations.

1. Make sure that you have all the roots of the equation you are solving and that each root obtained is actually a root of the original equation. (You may have lost a root or you may have introduced an extraneous root.)

2. If you multiply an equation by an expression containing the unknown, you may introduce an . . . root.

3. If you divide the equation by any factor containing the unknown, set . . . equal to zero and retain the roots of the resulting equation.

4. If you extract the square roots of both members of an equation, use . . . square roots of one member.

5. Do not divide by zero.

6. Use only principal roots in checking.

Equations in Quadratic Form

1. Solve for x^2: $2x^4 + 7x^2 - 15 = 0$ or $2(x^2)^2 + 7(x^2) - 15 = 0$ and check the roots. If $x^2 = \frac{3}{2}$, then $x = \pm\sqrt{\frac{3}{2}}; \pm\frac{1}{2}\sqrt{?}$. Explain. Check by substitution in the original equation. If $x^2 = -5$, then $x = \pm\sqrt{-5}$. You will be able to check the imaginary roots if you will remember that $(\sqrt{-5})(\sqrt{-5}) = -5$ by the definition of square root, and that

$$(-\sqrt{-5})(-\sqrt{-5}) = +(-5) = -5.$$

2. Solve for x: ① $x^{\frac{2}{3}} - 5x^{\frac{1}{3}} - 24 = 0$.

Plan of solution: ② $(x^{\frac{1}{3}} - 8)(\quad) = 0$

③ $x^{\frac{1}{3}} = 8$, therefore $x = 512$. Find another value of x and check.

3. Solve: ① $(x-2)^2 + 2(x-2) - 15 = 0$.

Plan of solution: ② $(x - 2 + 5)(x - 2 - 3) = 0$ etc.

4. Solve: ① $x - 3 + \sqrt{x-3} - 20 = 0$

② $(\sqrt{x-3} + 5)(\sqrt{x-3} - 4) = 0$

③ $\sqrt{x-3} + 5 = 0$, or ④ $\sqrt{x-3} = -5$

Complete and check. Use only the principal square roots in checking.

An equation may be said to be in quadratic form if it contains but two powers of the unknown, and one of these powers is twice the other.

In dealing with equations in quadratic form, we are generalizing as we did in Chapter VI. Notice the kinds of complications which can occur. They are similar to those illustrated in Chapter VI. Similarity of form, or isomorphism, is a very useful mathematical idea.

5. Solve and check:

① $y - 8\sqrt{y+3} + 18 = 0$

② $y + 3 - 8\sqrt{y+3} + 15 = 0$ Why was the second term introduced? Is the equation now in quadratic form?

③ $(\sqrt{y+3} - 3)(\sqrt{y+3} - 5) = 0$

④ $\sqrt{y+3} = 3$ Complete and check.

6. $x - 3\sqrt{x+1} - 3 = 0.$

7. $m - 7\sqrt{m-2} + 10 = 0.$

*8. $x + 5 = 4\sqrt{x+1}.$

*9. $x - 2\sqrt{x-3} = 66.$

*10. $y + \sqrt{y + 2a} = 2 - 2a.$

Other Equations Containing Radicals

Equations which contain radicals and which cannot be solved by the method of the preceding paragraph are solved by arranging in convenient form and then squaring. Observe the cautions listed on page 285.

Solve and check:

1. ① $\sqrt{3x+1} - \sqrt{x-1} = 2$

② $3x + 1 - 2\sqrt{3x^2 - 2x - 1} + x - 1 = 4$ ① squared

③ $2x - 2 = \sqrt{3x^2 - 2x - 1}$

Square ③ and solve the resulting quadratic.

2. $\sqrt{2x-5} - 3 = 0.$

3. $2\sqrt{6x-6} = 7 + \sqrt{3x+4}.$

4. $\dfrac{9}{\sqrt{2x-7}} = \sqrt{2x-7}.$

Try Exercise 57, page 463.

Part II. The Relation Between the Constants and the Roots in Quadratic Equations

Introduction

We are to investigate certain relations that exist between the constants a, b, and c and the roots r_1 and r_2 of the equation $ax^2 + bx + c = 0$.† As usual in the study of algebra, we are to seek general laws, to state them, and to apply them. We have already discovered that if a, b, and c are known, we can find r_1 and r_2, and we shall proceed to find answers to the following questions: (1) If the roots are known, can we find the constants, and hence an equation which has these roots? (2) If the constants of an equation are known, can we tell the character of the roots without actually solving the equation; that is, can we tell whether the roots are real or imaginary?

Finding the Constants when the Roots are Known

1. Consider an equation which has roots 6 and -7.

If $x = 6$, then $x - 6 = 0$, and if $x = -7$, then $x + 7 = 0$.

The expression $(x - 6)(x + 7)$ will equal zero if $x = 6$ and if $x = -7$, hence the equation $(x - 6)(x + 7) = 0$, or $x^2 + x - 42 = 0$, has the roots 6 and -7. Notice that the coefficient of x is the *sum* of the roots (6 and -7) with the sign changed, and that the constant term is the *product* of these roots.

2. Write an equation of which the roots are 5 and -9. Use the facts above.

3. Write an equation of which the roots are -2 and -7.

4. Let us state our conclusions in general symbols. In the equation $ax^2 + bx + c = 0$, $\quad r_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and

† We consider only those cases in which a, b, and c are real and rational.

$r_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$. Dividing the equation by a gives $x^2 + \dfrac{bx}{a} + \dfrac{c}{a} = 0$. Prove $r_1 + r_2 = -\dfrac{b}{a}$.

Plan of proof: Add the values of r_1 and r_2.

In a similar manner prove that $r_1 r_2 = \dfrac{c}{a}$.

Complete the following statement: In a quadratic equation in which the coefficient of x^2 is 1, the coefficient of x is equal to . . . and the constant term is equal to . . . Show that both of these statements apply to the equation $6x^2 - 5x - 4 = 0$, or $x^2 - \frac{5}{6}x - \frac{2}{3} = 0$, in which the roots are $\frac{4}{3}$ and $-\frac{1}{2}$.

5. Form an equation which has the roots 3 and $-\frac{1}{2}$, using the method suggested in the preceding example.

6. Solve the equation $x^2 - 5x + 6 = 0$ and check the roots by means of the theorems of Exercise 4.

The Character of the Roots

1. Consider the roots $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ of the equation $ax^2 + bx + c = 0$, in which a, b, and c are real and rational and $a \neq 0$. The radical may be real or imaginary, rational or irrational; the rest of the expression is real and rational. Therefore the radical determines the nature of the roots. Explain. $b^2 - 4ac$ is called the *discriminant* of a quadratic equation. Explain the significance of the term. For what value of $b^2 - 4ac$ are the roots equal? Are equal roots real or imaginary? For what values of the discriminant are the roots imaginary?

Explain and illustrate each of the following statements:

If $b^2 - 4ac < 0$, the roots are imaginary.

If $b^2 - 4ac = 0$, the roots are real and equal.

If $b^2 - 4ac > 0$, the roots are real and unequal. In this last case, if $b^2 - 4ac$ is a perfect square, the roots are rational; otherwise they are irrational.

You need not memorize these statements but you should be able to reproduce them at any time because you understand the reasons for them.

2. Determine without solving the character of the roots of the equations: (*a*) $x^2 + x - 30 = 0$

(*b*) $x^2 - 10x + 25 = 0$ (*c*) $x^2 - 10x + 24 = 0$

(*d*) $x^2 - 10x + 26 = 0$ (*e*) $20 - 10x - x^2 = 0$

Try Exercise 58, page 465.

Part III. Systems of Equations Involving Quadratics

The Linear-Quadratic Pair

1. Solve and check the following pair of equations:

$$① \ x^2 - 2xy - y^2 = 31 \text{ and } ② \ x + y = 13$$

Plan of solution: In the linear equation find a value for one of the unknowns in terms of the other and substitute it in the quadratic equation. ③ $x = 13 - y$, therefore

$$④ \ (13 - y)^2 - 2y(13 - y) - y^2 = 31$$

Solve this quadratic and substitute both its roots in ③. Record the results properly in pairs and check them.

Observe the cautions listed on page 285. All solvable linear-quadratic pairs can be solved by substitution.

2. Solve and check: ① $x^2 + y^2 = 13$, and ② $x + y = 5$.

Plan: Solve by substitution, or subtract the square of ② from ①, double the result, and add it to ①, then extract the square root of both members of the resulting equation.

3. Solve and check: ① $x + y = 12$, and ② $xy = 35$.

Plan: Solve by substitution; or from the square of ① subtract four times ② and extract the square root of both members of the resulting equation.

4. Discuss the graphic solution of the linear-quadratic pair and illustrate your discussion by reference to a graph in the text or to one that you have drawn yourself. Without solving examples 2 and 3 graphically, discuss their graphic solution.

Try Exercise 59, A, page 466.

The Quadratic Pair

***1.** What is the general form of quadratic equation in two unknowns? (See page 259.) The algebraic solution of two such general equations is beyond the scope of this course. The solution of so simple a pair as $x^2 + y = 7$ and $x + y^2 = 2$ leads to equations of the fourth degree which we are obviously not yet prepared to solve. For a few special cases, however, solution is possible by special methods which are available in this course, but any attempt to pursue the study far leads to a hunt for ingenious devices for special cases and is of little value. Graphic solution of this pair of equations is possible. Discuss it with reference to illustrations in Chapter VIII. The three following algebraic solutions may be of interest.

***2.** ① $x^2 + y^2 = 41$ and ② $xy = -20$

Plan of solution: Substitution.

③ $x = -\dfrac{20}{y}$	② $\div\ y$
④ $\dfrac{400}{y^2} + y^2 = 41$	③ in ①
⑤ $400 + y^4 = 41\,y^2$	④ $\times\ y^2$

⑥ $(y^2 - 25)(y^2 - 16) = 0.$ Explain and complete.

Complete and check.

3. ① $2x^2 - 3y^2 = 6$ and ② $x^2 - 2y^2 = 1$

③ ② $\times 2$

Plan of solution: Solve for x^2 and y^2 by combination, as linear pairs are solved. Be sure to list all four sets of answers properly paired.

4. ① $x^2 + y^2 = 10$ and ② $xy = -3$.

③ $3x^2 + 3y^2 = 30$ ① $\times 3$

④ $10xy = -30$ ② $\times 10$

⑤ $3x^2 + 10xy + 3y^2 = 0$ ③ + ④

⑥ $(3x + y)(x + 3y) = 0$

Set each binomial equal to zero and pair each resulting equation with ②.

Complete and check.

For other plans of solution see page 342.

Try Exercise 59, B, page 467.

Problems

Review the suggestions for problem solution on page 169.

In solving the following problems, use one or two unknowns and linear or quadratic equations, as you choose. In solving any problem of this course, if the equation is a quadratic with imaginary roots, either a mistake has been made or the problem is incapable of solution. If the roots are irrational, they should be determined to as many significant figures as the data permit or as the problem directs. If the roots are equal, the problem has but one solution. If unequal, it is quite possible that only one of them will check in the conditions of the problem. Discuss the answers to the following problems with these facts in mind.

1. A square tank 7 ft. high has 160 sq. ft. more in its four sides than in its bottom. How large is the bottom?

2. Solve the preceding problem for 2 ft. and 16 sq. ft. instead of 7 ft. and 160 sq. ft. respectively.

3. In Problem 1, change "more" to "less" and solve. Work to three-figure accuracy. Discuss the roots and the answer.

4. In Problem 1, change 160 sq. ft. to 200 sq. ft. and try to solve. Explain the result.

5. A sheepman bought sheep for $210, and after he had lost 5 by sickness, sold what he had left for $150, a loss of $1 a head. What did he pay for each sheep?

6. Are there two numbers which are reciprocals and the sum of which is $\frac{1}{2}$?

7. Two steamships pass each other, the first traveling directly north at 16 knots, the second east at 12 knots. Four and one half hours later the second sends out an S.O.S. call for help. How long will it take the first ship to reach the second?

***8.** Solve Problem 7 if the time is 6 hours instead of $4\frac{1}{2}$ hours.

***9.** The sum of the areas of two similar triangles is 255 sq. in. The ratio of their sides is 1:4. Find the area of each triangle. (The areas of any two similar triangles are to each other as the squares on any two corresponding sides.)

***10.** A merchant has a piece of carpet 15 ft. wide. How long a piece must he cut off if he wishes to divide the piece into three sections which shall have the same shape as the large one?

***11.** In order to determine the depth of a mine shaft, a man drops a stone into it, and $4\frac{1}{2}$ seconds later hears the sound of the stone striking bottom. If sound travels 1086 ft. per sec., and the distance, S, that a body falls from rest is given by formula $S = 16\,t^2$, how deep is the shaft?

***12.** The sum of the squares of two numbers is 61, and the product of the numbers is 30. What are they?

13. Can you find two numbers of which the sum is 13, such that the sum of their squares exceeds their product by 61?

14. A 13-foot ladder leaning against a wall lacks 3 ft. of reaching the top. A 17-foot ladder just reaches the top when its base is 3 ft. farther from the wall. How high is the wall?

***15.** At his usual rate a man can row 15 miles downstream in five hours less time than it takes him to return. If he could double his rate of rowing, he could row 15 miles downstream in one hour less than he could row back. What is his rate in still water, and what is the rate of the current?

Try Exercise 60, page 468.

Tests

Test A. On Chapter X

1. State four cautions to be observed in solving the equations of this chapter.

2. Prove the theorem: The sum of the roots of a quadratic equation in x, in which the coefficient of x^2 is 1, equals the coefficient of x with the sign changed.

3. State and prove a theorem about the product of the roots of a quadratic equation.

4. Give numerical illustrations of the theorems of 2 and 3.

5. Test the following pairs of numbers to see if they are roots of the equations opposite them. Use the theorems of the preceding examples.

(*a*) $3, -2$ $\qquad$ $x^2 + x - 6 = 0$

(*b*) $-3, 2$ $\qquad$ $x^2 + x - 6 = 0$

(c) $\frac{1}{2}$, $\frac{1}{3}$ $\qquad 6x^2 - 5x + 1 = 0$

(d) $-\frac{2}{5}$, $-\frac{3}{5}$ $\qquad 25x^2 + x - 6 = 0$

(e) $a + b$, $a - b$ $\qquad x^2 - 2ax - b^2 + a^2 = 0.$

6. What values of x will make the expression $x^2 - 3x - 19$ equal to 0? Express the answers in radical form and check.

7. Solve the equation $x^2 - 7x - 19 = 0$. Give the roots to two-figure accuracy. Express the roots in radical form and check in two ways.

8. Form an equation of which the roots are $-\sqrt{3}$ and $+\sqrt{3}$. Tell how you can recognize an equation in which the roots differ in sign only.

9. In the equation $H = \dfrac{2Pp}{P+p}$, let $H = 12$, $P = x + 3$, and $p = x - 2$ and find the value of x.

10. Make a summary of the important facts of this chapter.

Test B. Review Test

1. A chemist in speaking of the algebra he used in his daily work mentioned the three following examples:

(a) If $P_1 - P_2 = \dfrac{P_2 C_v}{C_p - C_v}$, find the value of $\dfrac{C_v}{C_p}$ in terms of the other letters.

(b) Is $\left(\dfrac{2x}{1+x}\right)^2 \div \dfrac{1-x}{1+x} \equiv \dfrac{4x^2}{1-x^2}$? Show how you make your decision.

*(c) Solve for x: $k = \frac{1}{4}x^{-2}(a - x)(b - x)$.

2. State and prove the formulas for A.P.

3. State and prove the formulas for G.P.

4. If an airplane could climb steadily at an angle of 22.00° with the horizontal and maintain a speed of 55.00 miles an hour, how high would it climb in half an hour? Answer the same question if the angle is reduced to 16.00° and the speed increased to 67.40 miles an hour.

5. (*a*) Solve and check: $\dfrac{x}{x+3} + \dfrac{6}{3x+2} = 1$

(*b*) Add $\dfrac{a}{a+3} + \dfrac{6}{3a+2}$

(*c*) Discuss fully the treatment of the denominators in (*a*) and (*b*).

***6.** Solve and check: $\dfrac{1}{x-2a} - \dfrac{1}{6x+a} = \dfrac{7}{3x-8a} - \dfrac{3}{2x-3a}$

Plan of solution: Unite the first two fractions and then the last two; then divide by $5x + 3a$. (See caution 3, page 285.)

***7.** A man blasting rocks set a fuse at a rock so that the explosion would occur in $\frac{2}{3}$ of a minute. He then ran away from the rock at the rate of 18 feet a second. How far could he run before he heard the explosion if sound travels 1080 feet a second?

8. Solve and check: $\dfrac{\sqrt{12+a}}{5} = \dfrac{3}{2+\sqrt{12+a}}$

CHAPTER XI

THE FORMULA FOR BINOMIAL EXPANSION

Introduction

1. This study deals with the expansion of a power of a binomial. Such an expansion is useful in the study of compound interest, for example, where such expressions as $(1.04)^n$ or $(1 + .04)^n$ are used.

The method of study which we are to use is that of observing a series of particular cases and from them drawing a general conclusion. This method is called *induction*. It is a very fruitful means of scientific investigation. Your ability to make such mathematical investigations, increased by the practice of the earlier work of this course, should enable you quickly to master the new formula and its applications.

2. The investigation. Supply the terms missing in the following identities. Perform the multiplications if necessary.

$$(a + b)^1 = a + b \qquad (1 + 0.04)^2 = 1^2 + 2(1)(0.04) + (0.04)^2 +$$
$$(a + b)^2 = a^2 + 2\,ab + b^2 \qquad (1 + 0.04)^3 = 1^3 + 3(1)^2(0.04) + 3(1)(0.04)^2$$
$$(a + b)^3 = a^3 + 3\,a^2b + ? \qquad \equiv 1 + 0.12 + 0.0048 + 0.000064 +$$
$$(a + b)^4 = a^4 + 4\,a^3b + ?$$

Let us now attempt to expand $(a + b)^5$ without resorting to multiplication. We may expect the first term to be a^5. Why? The last term will be b^5. Each remaining term will contain both a and b. The exponents of a form an A.P. in which $d = -1$, and the exponents of b an A.P. in which $d = 1$. The law for the coefficients is not quite so easy to discover, but a little study will show that the coefficient of each term after the first may be obtained from the preceding term by multiplying its coefficient

by the exponent of a and dividing by one more than the exponent of b.†

Expand:

3. $(a+b)^6$. **4.** $(a+b)^7$.

5. $(a+b)^8$. **6.** $(1+0.04)^4$.

7. $(1+0.006)^4 = 1 + 4(0.006) + 6(0.006)^2 + \dots$

If we disregard the remaining terms we have the result 1.024 which is correct to four figures. Comment on the rapid decrease in the value of the succeeding terms.

8. Make an investigation of the expansion of $(a-b)^n$ from $n=1$ to $n=8$ similar to that of **2**.

Expand:

9. $(a-c)^3$. **10.** $(1-b)^4$. **11.** $(\frac{1}{2}a - \frac{1}{2}b)^6$.

12. $(a-2)^5 = a^5 + 5a^4(-2) + 10a^3(-2)^2 + \dots$

13. $(1-0.05)^5$. **14.** $(2a-3b)^6 = (2a)^6 + \dots$

15. The binomial formula. Study the following expansion; explain each term and write at least two more terms:

$$(a-b)^n = a^n - na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}a^{n-4}b^4 - \dots$$

In each term what is the sum of the exponents of a and b?

† An interesting solution of the problem of finding these coefficients was made by Pascal, who arranged the consecutive sets of coefficients in a triangle in which each term is the sum of the two terms above it.

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

Observe that the coefficients in any binomial expansion form a series of integers beginning with 1, first increasing, and then decreasing by the same steps until unity is reached again. This symmetry is remarkable.

***16. Abbreviating the coefficients in the binomial formula.** The formula may be somewhat abbreviated by the use of a new symbol. 4! is read "factorial 4" and means $1 \cdot 2 \cdot 3 \cdot 4$. Show that 3! and 4! may be used in writing the third and fourth terms respectively. Write 5 terms of the expansion of $(a - c)^9$ using the factorial notation in the denominators.

***17.** With the factorial notation it is possible to write the formula in still more compact form. Consider the sixth term of $(a - b)^9$ which is $-\dfrac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} a^4b^5$ or $-\dfrac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{5!} a^4b^5$. Multiplication of numerator and denominator by 4! gives $-\dfrac{9!}{5!\,4!}a^4b^5$. This is a form which has two excellencies which mathematicians seek in their formulas; namely, compactness and symmetry. It is a form easy to remember. Write in this form the expansion of $(b - x)^8$, $(c - 3d)^7$, $(a - 1)^6$, and $(2 - b)^5$.

Try Exercise 61, A, page 469.

18. To find any term. Study the terms of the formula and of the preceding illustrations in order to discover a method of writing any term of a binomial expansion without writing the preceding term. How may the sign be determined? the exponent of b? the exponent of a? The denominator of the coefficient? the numerator? Test your conclusions by writing the fifth term of $(a - b)^8$. If necessary, verify the result by writing the four preceding terms. Observe that the exponent of b is one less than the number of the term and that this may be considered the key number for the term.

Write the seventh term of $(a + b)^{15}$. The key number is 7–1 or 6. This is the exponent of b. It is also the number of factors in the numerator and in the denominator. The exponent of a is 15 minus this key number.

Write the following terms:

19. The fifth term of $(a - b)^9$.

20. The tenth term of $(a - 3b)^{20}$.

21. The thirty-second term of $(a - 5b)^{85}$.

22. The fourth term of $\left(\frac{a}{b} - \frac{b}{a}\right)^7$.

Try Exercise 61, B, page 470.

Negative and fractional exponents. With certain restrictions which it is not worth while to discuss here, the binomial formula may be used for negative and for fractional values of n. It will be sufficient for the purposes of this course if the pupil understands the expansions which follow.

***1.** The first five terms of $(1+b)^{-1}$ are $1^{-1} - 1(1)^{-2}(b) + 1(1)^{-3}b^2 - 1(1)^{-4}b^3 + 1(1)^{-5}b^4$. Write these terms in simpler form and verify them by writing $(1+b)^{-1} \equiv \frac{1}{1+b}$ and performing the division indicated.

***2.** $$\begin{aligned}\sqrt{10} = \sqrt{9+1} = \sqrt{9(1+\tfrac{1}{9})} &= 3(1+\tfrac{1}{9})^{\frac{1}{2}}\\ &= 3[1^{\frac{1}{2}} + \tfrac{1}{2}(1)^{-\frac{1}{2}}(\tfrac{1}{9}) - \tfrac{1}{8}(1)^{-\frac{3}{2}}(\tfrac{1}{9})^2 + \ldots]\\ &= 3(1 + \tfrac{1}{18} - \tfrac{1}{8}\cdot\tfrac{1}{81} + \ldots) = 3.1623.\end{aligned}$$

***3.** $$\begin{aligned}\sqrt[3]{10} = \sqrt[3]{8+2} = \sqrt[3]{8(1+\tfrac{1}{4})} &= 2(1+\tfrac{1}{4})^{\frac{1}{3}}\\ &= 2[1 + \tfrac{1}{3}(1)^{-\frac{2}{3}}(\tfrac{1}{4}) - \tfrac{1}{9}(1)^{-\frac{5}{3}}(\tfrac{1}{4})^2 + \ldots]\\ &= 2(1 + \tfrac{1}{3}\cdot\tfrac{1}{4} - \tfrac{1}{9}\cdot\tfrac{1}{16} + \ldots) = 2.1547\end{aligned}$$

***4.** $$\begin{aligned}\sqrt[4]{1.006} &= (1+0.006)^{\frac{1}{4}}\\ &= 1 + \tfrac{1}{4}(1)^{-\frac{3}{4}}(0.006) - \tfrac{3}{32}(1)^{-\frac{7}{4}}(0.006)^2 + \ldots\\ &= 1 + \tfrac{1}{4}\times 0.006 - \tfrac{3}{32}\times 0.006^2 + \ldots\end{aligned}$$

The sum of the first two terms is 1.0015 which is correct to

five-figure accuracy. The third term and all that follow are so small as to be negligible unless more than five-figure accuracy is necessary.

Try Exercise 61, C, page 471.

An Application of the Binomial Formula

1. The compound interest formula $A = p(1 + r)^n$ lends itself to the use of the binomial formula as is illustrated in the following example.

How much will \$400 amount to in 4 years if interest is compounded annually at 5%?

Plan of solution: Simple interest for 4 years at 5% would amount to \$80, hence the amount will be approximately \$480. The answer, then, to the nearest dollar is expressible in three figures; to the nearest cent in 5 figures. Let us work to six-figure accuracy and then round off the result to five figures.

$$A = 400(1 + 0.05)^4 = 400[1^4 + 4(1)^3(0.05) + 6(1)^2(0.05)^2 + 4(1)(0.05)^3 + (0.05)^4] = 400(1 + 0.20 + 0.015 + 0.0005 + 0.0000625) = 400(1.2155625) \text{ or } 400(1.21556) = \$486.224.$$

Verify as much of this result as is possible by the use of the tables of logarithms. Notice that in these computations the use of logarithms is usually less laborious than the use of the binomial formula. It is, however, important to understand such uses of the binomial formula because of their value in the study of the theory of investment, etc.

2. Find the amount of \$220 at 4% compounded quarterly for 10 years.

Plan of solution: (Estimate the result as was done in example 1.)

$$\begin{aligned} A &= 220(1.01)^{40} \\ &\equiv 220[1 + 40(1)^{39}(.01) + 780(1)^{38}(.01)^2 + 9880(1)^{37}(.01)^3 \\ &\quad + 91390(1)^{36}(.01)^4] \\ &\equiv 220(1 + .4 + .078 + .00988 + .0009139) \\ &\equiv 220(1.48879) \equiv ? \end{aligned}$$

Notice that the remaining terms do not affect the answer when it is expressed to the nearest cent. Notice that such interest problems can be done with very little use of pencil.

Try Exercise 61, D, page 472.

Tests

Test A

1. Expand to five terms $(x+y)^n$.

2. Expand completely $(a-\frac{1}{2}b)^5$.

3. Find and simplify the 8th term of $(2x-y)^{12}$.

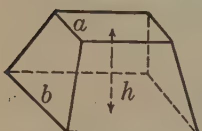
4. Find without logs the interest on \$100 at 4% compounded semi-annually for five years.

5. Find the positive root of the following equation to the nearest hundredth: $x^2-5x-10=0$.

6. How many ounces of a 20% solution of a certain acid must be added to 20 ounces of a 10% solution to make a new solution that shall contain 12% of the acid?

Test B

***1.** The formula for the volume of a frustum of a square pyramid is $V=\frac{1}{3}h(a^2+ab+b^2)$, where a, b, and h are the dimensions shown in the accompanying figure. Find the area of the lower base of such a frustum if it has an altitude of 12.50 inches, an upper base area of 39.69 sq. in., and a volume of 2000 cu. in.



2. Solve graphically, and check by substituting the roots in both equations:

$$3x+y=5 \qquad x^2+x-3y=9$$

3. Write an equation which shall have for its roots $2 \pm \sqrt{3}$. Prove that the equation which you have written has these roots.

4. Solve and check:

$$2x + \tfrac{2}{3}(x - 5y) = \frac{y - x}{4} - 7\tfrac{5}{8} \qquad x - y + 2\tfrac{1}{2} = 0$$

5. Find without logarithms the interest on \$3000 at 4% compounded quarterly for one and one fourth years.

6. Use logarithms to find the interest for 12 years on \$3280 at 5% compounded semi-annually.

7. Expand, and simplify each term: $\left(\dfrac{2x}{3} - \dfrac{3y}{2}\right)^6$

***8.** Find the middle term of $(a + 3b)^6$. Find the term of $(a^2 + a)^{11}$ that contains a^{15}.

***9.** A tinsmith has a sheet of tin 13 in. sq. from which he must make a box that shall contain 147 cu. in. What is the size of the square which he must cut from each corner of his tin?

10. A man received \$184.05 as two years compounded interest on \$2000. What per cent interest did he receive on his money?

11. By means of logarithms, divide 4.237 by .02458; and extract the cube root of .02456.

12. Solve and check: $a - \dfrac{a^2}{ax} = ab$

13. Solve and check: $x = -y - 2 \qquad x^2 + xy + y^2 = 19$

CHAPTER XII

SUMMING UP THE COURSE

Your study of the Second Course in Algebra has given you new information, new mathematical power, and new insight into how man thinks out many of his problems; it has given you many illustrations of that important method of work called the scientific method. In order to make the course as profitable as possible, you should now sum up its important ideas and bring them all together in your mind. Consider the following plans for doing this, and as you study them, remember that the ability to organize ideas is important for you no matter what course you are to pursue in the future.

1. Make a brief outline of each chapter. Mention the most interesting and most important ideas contained in it.

2. Answer the introductory questions at the beginning of each chapter or part of a chapter.

3. Make a test covering the work of each chapter.

Preparation for Examinations

If you are to be examined on the work of the course, you need to have at your finger-tips certain facts, formulas, and methods. The following exercises have proved helpful for this purpose.

1. In each exercise of Part II (pages 347 and following) work sample examples. Do, for instance, **4, 8, 12,** etc., in each set. For an easy review, work the earlier examples in each set; for a harder review, the later ones, or the starred ones. If any exercise proves difficult for you, do enough of the examples in that exercise to clear up the difficulty; study if necessary the text referred to at the head of the exercise.

2. Solve the verbal problems on pages 185 and following and on pages 236 and following.

3. Repeat the "Chapter Tests" at the end of each chapter.

4. Do again, carefully, the "Review Tests" or "Cumulative Tests" that appear throughout the book. See particularly pages 20–23, 52–54, 72–76, 91–95, 108–111, 136–140, 164–167, 194–200, and the tests of Chapter XII.

Give the meaning of each formula below. Derive each formula and check your work by comparing it with the derivation given in the book.

Exponents: **5.** $a^b \times a^c \equiv a^{b+c}$ **6.** $\dfrac{a^b}{a^c} \equiv a^{b-c}$

7. $(a^b)^c \equiv a^{bc}$ **8.** $\sqrt[c]{a^b} \equiv a^{\frac{b}{c}}$

Logarithms: (Before attempting the proofs, give a clear definition of a logarithm).

***9.** $\log MN = \log M + \log N$ ***10.** $\log \dfrac{M}{N} = \log M - \log N$

***11.** $\log M^p = p \log M$ ***12.** $\log \sqrt[r]{M} = \dfrac{1}{r} \log M$

Arithmetic progression: **13.** $l = a + (n-1)d$

14. $s = \dfrac{n}{2}(a + l)$ or $s = \dfrac{n}{2}[2\,a + (n-1)d]$

Geometric progression: **15.** $l = ar^{n-1}$

16. $s = \dfrac{ar^n - a}{r - 1}$

17. Sum "to infinity": $s = \dfrac{a}{1-r}$

Quadratic equations: **18.** $x = \dfrac{-b \pm \sqrt{b^2 - 4\,ac}}{2\,a}$

19. $r_1 + r_2 = -\frac{b}{a}$

20. $r_1 r_2 = \frac{c}{a}$

Binomial Expansion: **21.** $(a \pm b)^n =$

$$a^n \pm na^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 \pm \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 + \dots$$

Test on Chapter II

1. Define the sine, cosine, and tangent of an acute angle.

2. Show that when A and B are the acute angles of a right triangle, $\sin A = \cos B$ and $\cos A = \sin B$.

3. The sine of a certain angle is $\frac{5}{13}$. Find its cosine and tangent without using the tables.

4. The cosine of a certain angle is $\frac{84}{85}$. Find its sine and tangent without using the tables.

5. The tangent of a certain angle is $\frac{11}{60}$. Find its sine and cosine without using the tables.

6. In any 30°, 60°, 90° triangle the shorter leg is half as long as the hypotenuse. Draw such a triangle and find: (*a*) the sine, cosine, and tangent of 60°; (*b*) the sine, cosine, and tangent of 30°. Leave the results in radical form.

7. Solve the right triangle in which $c = 112.0'$ and $A = 32° 45'$.

8. Solve the right triangle in which $a = 10.5''$ and $b = 7.00''$.

9. Solve the right triangle in which $b = 56.6'$ and $A = 45.8°$.

10. In order to find the distance, CB, across a ravine, a base line CA, 60.8 ft. long, is measured along one edge from C and at a right angle to BC; and the angle BAC is measured and found to be 40.5°. Find CB.

11. A straight toboggan slide, built on level ground, is to start from a platform 40 ft. high and meet the ground at an

angle of 12.5°. How long will the slide be? Answer to the nearest foot.

*12. When the elevation of the sun is 65° 30′, a tower casts on level ground a shadow 52.0 ft. long. How high is the tower?

*13. A ship passes a buoy 4200 yards off shore, and sails for fifteen minutes parallel to the shore. During this time it has moved through an angle of 52.3° as seen from the point on the (straight) shore which is nearest to the buoy. Find to the nearest mile per hour the speed at which the ship sails.

Test on Chapter III

1. Explain and illustrate the difference between a coefficient and an exponent.

2. Perform the operations indicated:

$$a^5 \cdot a^{-4} \qquad a^4 \div a^7 \qquad a^7 \div a^{-9} \qquad \sqrt{a^6} \qquad \sqrt[3]{a^{12}}$$
$$\sqrt[4]{9} \qquad (a^2)^3 \qquad (a^3)^{-2}$$

3. Show why $a^0 = 1$ and that $a^{-n} = \dfrac{1}{a^n}$. Does $3\,a^{-2} \equiv \dfrac{1}{3\,a^2}$?

Simplify:

4. $\sqrt[3]{\dfrac{18}{3^5}}$ 5. $(a^{-2}b^{\frac{1}{2}}c^{-\frac{3}{4}})^{-6}$ 6. $\left(\dfrac{\sqrt[3]{3^{\frac{1}{3}}}}{\sqrt[3]{2^{-2}}}\right)^{-6}$

7. $\left(\dfrac{3\,a^{-2}b^2}{2\,x^{\frac{4}{5}}}\right)^5$ *8. $\left(\dfrac{-216\,a^6}{64\,c^9d^{6m}}\right)^{\frac{1}{3}}$

*9. $\dfrac{a^{\frac{1}{2}}+b^{\frac{1}{2}}}{a^{\frac{1}{2}}-b^{\frac{1}{2}}}+\dfrac{a^{-\frac{1}{2}}+b^{-\frac{1}{2}}}{a^{-\frac{1}{2}}-b^{-\frac{1}{2}}}$ *10. $\left(\dfrac{\sqrt{m}}{n^{\frac{1}{2}}}\right)\left(\dfrac{\sqrt[5]{n^2}\cdot n^{\frac{3}{5}}}{\sqrt[3]{m^2}\cdot m^{-\frac{7}{8}}}\right)$

11. A very important number in the quantum theory of advanced physics is $h = .000\ 000\ 000\ 000\ 000\ 000\ 000\ 000\ 00655$. Write it in a more convenient form.

12. Define logarithm and illustrate.

13. Supply the missing numbers:

Number...	16	125	?	?	9	64	$\frac{1}{81}$	.1
Base......	2	5	5	4	?	?	3	100
Logarithm.	?	?	2	3	2	3	?	?

14. Simplify: $\log_4 16 + \log_3 9 + \log_{10} 100 + \log_3 1$

15. Simplify: $3 \log_3 9 + \log_{10} .01 + \log_3 \frac{1}{9}$

16. Simplify: $\frac{1}{2} \log_3 9 - 2 \log_{27} 3 + \log_2 2 - \log_2 1$

Use logarithms in the computations of the examples below:

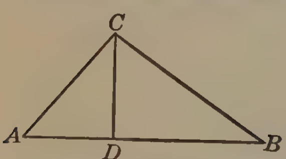
17. A formula for the area of a triangle is $k = \frac{1}{2}\, ac \sin B$. Find the area of a triangle if the two sides are respectively 127 in. and 129 in., and the included angle is 61.5°.

18. A formula for the area of a triangle is $k = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$. Find the area of a triangle with sides 18.27 in., 25.36 in., and 15.49 in.

19. Estimate the value of this fraction and then compute it.

$$\frac{23.10\sqrt{8.234}}{\frac{1}{2}\cos 24°29'}$$

20. In the accompanying triangle, $AC = 21.35$ ft., angle $A = 46°31'$, $B = 24°29'$, and angle $ACB = 109.0°$ Find BC and CD.



21. The volume of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$. What radius must be used in constructing a sphere of volume 75.34 cu. in.?

Test on Chapter IV

1. How is any term determined from the term preceding it in an A.P.? In a G.P.?

2. Given the formulas $l = a + (n - 1)d$ and $s = \frac{n}{2}(a + l)$, eliminate a and solve the resulting formula for d.

3. Find by formula the 12th term and the sum of 12 terms of $\frac{1}{2} + 1 + 2 + \ldots$

4. The fourth term of an A.P. is $\frac{3}{2}$ and the 7th term is 0. Write the first three terms.

5. Of how many terms of the progression $1\frac{1}{2} + 3 + 4\frac{1}{2} + \ldots$ is $1912\frac{1}{2}$ one half the sum?

6. Find the sum of the first 20 odd numbers. Of the first 20 numbers divisible by 5.

7. Given the terms 3 and $\frac{3}{8}$: Insert two geometric means. Find the sum of the resulting series to 7 terms. To infinity.

8. Insert 3 arithmetic means between a and l.

9. The sum of the first 4 terms of an A.P. is 30. The first, third, and ninth terms form a G.P. Find the progression.

***10.** Find the value at the end of three years of $600 at 6% interest compounded semi-annually.

***11.** Three numbers form an A.P. with a common difference of -2. If the first is reduced by 2 and the second doubled, the three numbers form a G.P. Find the original numbers.

***12.** The sum of 7 terms of a progression which has the common ratio 2, is 254. Find the last term.

13. Derive a formula for a sum to infinity.

14. Find the limiting value of 8.2121...

***15.** A rubber ball dropped from a height of 24 ft. rebounds half that distance and continues to rebound in this way after each descent. If it continues to rebound how far will it have

traveled when it touches the ground for the sixth time? When it comes to rest?

16. Find the sum of all three-digit numbers that are divisible by 11.

***17.** Find the 15th term and the sum of 10 terms of

$$2, \quad 4+\sqrt{2}, \quad 6+2\sqrt{2} \ldots$$

***18.** A man buys a phonograph for $400, paying $40 down and $40 a month. The interest on the money due is 6%. What will the phonograph actually cost him?

***19.** An automobile costing $2400, depreciates at the rate of 40% per year. At this rate when will it be worthless? How much will it be worth in 5 years?

Test on Chapter V

1. Find the average speed at which a man must make an 180-mile auto trip in order that a 10-mile per hour reduction in the average rate will make the trip take him only 3 hours longer.

2. A dealer has coffee worth 30 cents a pound and coffee worth 40 cents a pound. In what proportion must he mix them in order that the mixture may be worth 36 cents a pound?

3. At a recent industrial exposition the admission prices were 75 cents for adults and 25 cents for children. On a certain day the turnstiles showed that 4468 persons had paid admission, and the box office receipts were $2751. How many adults paid admission on the day in question?

4. A tank can be filled by one pipe in 24 minutes or by a second pipe in 30 minutes. It can be emptied by an exhaust pipe in 20 minutes. If all three pipes are open, will the water level in the tank rise, fall, or remain stationary? Answer the question below which is consistent with your answer. If the tank is empty and all three pipes are open, how long will it take

to fill the tank? If the tank is full and all three pipes are open, how long will it take to empty the tank?

5. Working together on a certain job, two men complete it in 4 days. On a similar job the first man works alone for 3 days and then is joined by the second man. They finish the job in another 3 days. The foreman has 10 more exactly similar jobs, all to be done by the first man. How much time should he allow for the ten?

***6.** A goldsmith has a 40 oz. alloy of gold and silver. To find how much of it is gold he weighs the alloy in water and finds that it loses 2.5 oz. His tables show him that gold loses 1 oz. in 19 and silver 1 oz. in 10 when weighed in water. How much of the alloy is gold?

7. A piece of carpet 20 ft. wide is to be cut into 3 equal sections each of which shall have the same shape as the original piece. How long must the original piece be?

8. A train runs 175 miles from A to B and arrives at B at 5:20. A new schedule is put into effect under which the train leaves A at the same time as before, but travels 5 miles per hour faster and reaches B at 4:10. Find both rates.

***9.** A local train runs 72 miles to a junction where it makes exact connections with an express. A new schedule requires the express to leave 16 minutes earlier and the local to leave its starting-point at the same time as before. It is desired to have the local arrive at the junction 9 minutes before the express leaves. This can be done if the local runs 5 miles per hour faster. What is the new rate of the local train?

10. A man receiving a certain legacy buys an automobile for \$1800 and invests one fourth of the rest of the money at 3% and the remainder at $4\frac{1}{2}$%, thereby getting an annual income of \$165. What was the amount of the legacy?

11. A room is two thirds as wide as it is long. A second room

is 1 ft. longer and 5 ft. narrower and has 148 sq. ft. less area. Find the dimensions of the two rooms.

12. The sum of two numbers is 19 and their difference is 9. What are the numbers?

13. A grocer estimated that his sugar supply would last him 8 weeks. He sold an average of 50 pounds a week more than he expected, and the sugar lasted 6 weeks. How much had he?

14. The telephone poles along a certain road are at equal intervals. If the intervals between the poles were increased by 22 ft., there would be 8 fewer in every mile. How many are there in a mile?

15. Together A and B have d dollars. A gives B n dollars and they then have equal amounts. How much has each?

16. A man invests $1000, part at 4% and the rest at 5%. The resulting income for two years is $96. How was the money divided?

17. Find two consecutive numbers which satisfy the following condition: 4 times the first minus three times the second equals 9.

18. A man invests $5650 in two parts, one part at 4% and the remainder at 6%. His annual income therefrom is $298. How much did he invest at each rate?

19. In a solution of 60 lb. of salt and water, 3 lb. are salt. How much water must be evaporated to leave a solution 10% salt?

***20.** For fencing and sodding a part of a back yard, $47.50 is to be spent. It is desired to have the length of the plot 15 ft. more than the width. For the sodding 12 cents a sq. ft. is to be paid and for the fencing, 75 cents a linear yard. How large a plot can be sodded and fenced?

Test on Chapter VI

Factor if possible:

1. $a^2 + 21a + 98$
2. $20xy - 28xw - 5yz + 7wz$
3. $x^2 + xy - 72y^2$
4. $x^2 - a^2 - b^2 - 2ab$
5. $6x^2 - 7x - 20$
6. $2mn + 2bn + 4m^2 + 4bm$
7. $x^4 - y^2 - 6y - 9$
8. $x^3 - 18x^2 - 40x$
9. $2an^2 - 32an + 126a$
10. $x^2 - 169$
11. $a^4b^4 + 2a^2b^2 + 1$
12. $2x^2 + 3x - 54$
13. $x^8 - 16y^8$
14. $a(a - c) - b(b - c)$
15. $a^2 + b^2$
16. $x^2 - xy + y^2$
17. $x^2 - 2xy - y^2$
18. $.2m^2 + .72mn - 1.4n^2$
19. $k^4 - k^2 + 12k - 36$

*20. $a^{3n} + b^{3n}$

*21. $m^3 - 7m^2 - 4m + 28$

22. $72 + 7x - 49x^2$

*23. $27(a - b)^3 + 8b^3$

24. $m(m + 1) - n(n - 2m)$

*25. $8m^3 - 27n^3$

26. $s(s + 1)(s + 3) - (2s + 6)$

*27. $125y^3 - 8z^3$

28. $m^3 + n^2 + m + n$

29. $6x^2 + 5x - 6$

30. $2a^2 + 3ax - 2x^2$

31. $y^2 - x^2 - 1 - 2x$

32. $12x^2yz^2 + 6x^2y^2z^3 - 9x^3y^3z$

33. $5b^3 - 26ab^2 + 5a^2b$

34. $\dfrac{a^2}{4b^2} + \dfrac{4b^2}{a^2} + 2$

35. $(a - 2)^2 - y^2$

36. $21 + 4x - x^2$

*37. $x^3(x + y) - y^3(x + y)$

*38. $a^2 - 28a + 2ax - 28x + x^2 + 192$

*39. $4m^2 - 9m - 8mn + 9n + 4n^2 + 5$

*40. $x^3 - (4x + 7x^2) + 28$

*41. $x^2 + xy - 4z^2 - 2yz$

*42. $5x^4 + 4x^3 - 20x - 125$

*43. $x^4 + 4x^3 - 12x - 9$

Test on Chapter VII

1. What is generally meant by "simplifying" a fraction?

2. What three things are meant in this course by "simplifying" a radical or an expression containing a radical?

Simplify:

3. $\dfrac{1}{m} - \dfrac{1}{3m}$

4. $\dfrac{2}{1-a} + \dfrac{1}{a-1}$

5. $\dfrac{7-3\sqrt{3}}{2+\sqrt{3}}$

6. $\dfrac{2+\sqrt{12}}{2}$

7. $\dfrac{3-3\sqrt{2}}{\sqrt{2}+1}$

8. $\dfrac{\dfrac{x}{y} - 4 - \dfrac{21y}{x}}{\dfrac{x}{y} + 1 - \dfrac{6y}{x}}$

9. $\dfrac{2+\sqrt{5}}{\sqrt{5}-1}$

***10.** $\dfrac{x - \dfrac{x^2}{x+y}}{x-y} + \dfrac{x^2+xy+y^2}{x^2-y^2}$

11. $\dfrac{x^2 + \dfrac{1}{x}}{x + \dfrac{1}{x} - 1}$

12. $\dfrac{c+v}{1+\dfrac{cv}{c^2}}$

13. $(n+v) \div \left(1 + \dfrac{nv}{c^2}\right)$

14. $\dfrac{\dfrac{1}{m} - \dfrac{m+n}{m^2+n^2}}{\dfrac{1}{n} - \dfrac{m+n}{m^2+n^2}}$

15. $\dfrac{\dfrac{1}{x} + \dfrac{4}{x^2} + \dfrac{4}{x^3}}{1 + \dfrac{5}{x} + \dfrac{6}{x^2}}$

16. $\dfrac{(m+1)^{\frac{1}{2}} - 2(m-1)^{-\frac{1}{2}}}{m-3}$

***17.** $\left(\dfrac{x^{\frac{1}{2}} + y^{\frac{1}{2}}}{x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y} \div \dfrac{x^{\frac{1}{2}} - y^{\frac{1}{2}}}{x-y}\right)\dfrac{1}{x-y}$

Solve and check:

18. $\dfrac{3}{x} + \dfrac{5}{2x} = 3$

19. $\dfrac{9}{x-7} - \dfrac{9}{x-2} = \dfrac{5}{x-8} - \dfrac{5}{x+1}$

20. $\dfrac{d^2}{x} - d = \dfrac{a^2}{x} + a$

21. $\sqrt{\dfrac{1}{x}} = 5$

22. $\dfrac{5x}{2} = 22 + \dfrac{7x}{5}$

23. $\dfrac{x - \frac{1}{2}}{x + \frac{1}{2}} - \frac{1}{2}\left(\dfrac{2x-3}{x-1}\right) = 0$

24. $\dfrac{17 + \frac{3}{x}}{3} + \dfrac{1 + \frac{18}{x}}{5} = \dfrac{\frac{21}{x} - 1}{9} + \dfrac{\frac{100}{x} + \frac{5}{3}}{15}$

25. $3 - \dfrac{1}{\frac{1}{3} + \frac{1}{x}} = \dfrac{1}{3}$

26. $\dfrac{y^2-3}{1-y^2} + \dfrac{1}{y-1} = \dfrac{4}{y+1} + \dfrac{y}{1-y}$

27. A boys' club was buying a boat for \$160 and dividing the expense equally. By taking in three new members they reduced the expense per boy by \$12. How many members had the club at first?

28. What must be the scheduled rate of a train in order that a reduction of 20 miles per hour shall make a 300-mile trip take 4 hours longer?

29. On a car leaving Boston for New York there were 2 more men than women. At Springfield 3 women left and at Hartford 3 men left and 4 men got on. There were then twice as many men as women. How many men and how many women were in the car when it left Boston?

*30. Two boys started at one corner of a rectangular field containing 30 acres and ran in opposite directions until they met at a point 28 rods from the opposite corner. If the rate of one boy is two thirds that of the other, what are the dimensions of the field?

31. Divide 72 into two parts in such a way that one part shall equal one fifth of the other.

32. The length of a rectangle is $5\frac{1}{2}$ ft. more than the width. If the length is decreased by 3 ft. and the width is increased by $1\frac{1}{2}$ ft., the area will be unchanged. Find the dimensions of the rectangle.

33. One man can do a job in $3\frac{1}{2}$ hours and another in 4 hours. If they both work at the job what time should be allowed?

***34.** Solve the preceding problem for a hours and b hours, and thereby obtain a formula for the solution of such problems.

***35.** The rate of a train running m miles per hour is increased by 5 miles per hour. How much time will be saved on a 200-mile run?

Test on Chapter VIII

1. The area of a circle is given by the formula $A = \pi r^2$. Use this formula to show what we mean when we say that the area "depends upon" or "is a function of" the radius. Complete the following sentence: The area of a circle varies as the . . . . the radius.

2. By what four methods can you represent the nature of the dependence of one number upon another number?

3. The intensity of illumination of an object varies inversely as the square of its distance from the source of light. State this fact by means of a formula.

4. Show by means of a table how the intensity of illumination received by the page of a book varies when the book is moved from a point 15 in. from the light to a point 25 in. from the light. Call the illumination at 15 in. 100 units; use only integral values for the distance.

5. Make a graph from the table of the preceding question.

6. Write a formula which shows the dependence of b upon a in the following table:

a	1	2	3	4...
b	5	9	13	17...

7. Plot the graph of $x^2 - 6x + 5 = y$. For what values of x is y negative? positive? zero?

***8.** What is the shape of the graph of each equation (9–14) below?

Solve graphically estimating the roots as closely as possible and checking them by substitution or otherwise:

9. $\dfrac{x+2}{3} - 4 = 0$ **10.** $x^2 + x - 12 = 0$

***11.** $x^3 - 2x^2 - 5x + 6 = 0$

12. $x + 7y = -19 \qquad 3x = -2y$

13. $x^2 + y^2 = 13 \qquad 2x - y = 4$

***14.** $x^2 + y^2 = 13 \qquad x^2 - y^2 = 5$

15. $x^2 - y^2 = 5 \qquad 2x - y = 4$

Test on Chapter IX

1. What is meant by saying that 5 and 3 are "roots" of simultaneous linear equations in x and y? How are the roots indicated graphically?

2. Is it possible to solve all pairs of linear equations in two unknowns? Illustrate your answer.

Solve the following systems of equations and check the results:

3. $x + 2y = 2 \qquad 3x - 4y = 21$

4. $x - 5y = 3 \qquad 3x + y + 23 = 0$

5. $5x - 7y = 21 \qquad x - 4y = -1$

6. $2x - 5y - 10 = 0 \qquad 9x + 8y - 14.5 = 0$

7. $3x + y = 2 \qquad .4x - 2.8y = 1$

8. $\frac{x}{3} + \frac{y}{4} = 6 \qquad \frac{x}{6} - \frac{y}{3} = 2\frac{1}{2}$

9. $x - \frac{11 - y}{3} = 18 \qquad 2y = 29 - \frac{x - 13}{4}$

***10.** $\frac{8}{x} - \frac{15}{y} = 33 \qquad \frac{4}{x} - \frac{35}{y} = -43$ Do not clear of fractions at the outset.

11. $\frac{9}{x} + \frac{4}{y} = 5 \qquad \frac{6}{x} - \frac{3}{y} = -2\frac{1}{3}$

***12.** $\frac{x + by}{3a + 2} = b \qquad \frac{2ax - by}{a} = b$

***13.** $x - \frac{2y + 1}{5} = 2 + 2(x - y) \qquad \frac{x}{3} - \frac{y}{2} = \frac{3(x + y + \frac{5}{2})}{4}$

14. $8x - 2y - 4z = 1 \qquad 8x + 7y + 15z = 0 \qquad x + y + z = \frac{9}{8}$

***15.** $x + 8y + 5z = 1 \qquad 3x + 4y + 10z = 1\frac{1}{8} \qquad x + 4z = 0$

Test on Chapter X

1. What is a "quadratic" equation in one unknown?

2. How many roots has a quadratic equation in one unknown? Illustrate your answer.

3. Solve by completing the square $2x^2 - 7x + 6 = 0$

4. Solve by completing the square $ax^2 + bx + c = 0$

5. Derive a formula for the sum of the roots of a quadratic equation in one unknown.

6. Derive a formula for the product of the roots of a quadratic equation in one unknown.

7. Translate into words the two preceding formulas.

8. Why is $b^2 - 4ac$ called the "discriminant" of a quadratic equation?

Solve the following equations; check by means of the relations between the constants and the roots.

9. $4x^2 - 29x + 7 = 0$
10. $2x^2 + 3x - 5 = 0$
11. $6x^2 + x - 2 = 0$
12. $8x^2 + 34x + 15 = 0$
13. $x^2 - 5x + 4 = 0$
14. $3x^2 - 5x - 6 = 0$
15. $\frac{x^2}{4} - \frac{3x}{2} - \frac{5}{2} = 0$
16. $x^2 - 2x - 1 = 0$
17. $ax^2 + 2ax - 3 = 0$
18. $2x^2 - 5nx = 3n^2$

*19. $x^2 - 8.12x - 33.109 = 0$

*20. $x^2 - 2.1x + 0.8371 = 0$

21. $8x^2 + 9x + 0.4 = 0$

*22. $x^2 + x + 2\sqrt{3} = 0$

23. $\sqrt{x+4} + x = 8$
24. $\sqrt{x^2 - 9} + 21 = x^2$
25. $4x^4 - 5x^2 + 1 = 0$

*26. $x^4 - 7x^2 - 18 = 0$

Without solving determine the nature of the roots of:

27. $4x^2 - 4x + 1 = 0$
28. $2x^2 - 3x = 5$
29. $x^2 - 5x = 75$
30. $x^2 + x + 2 = 0$
31. $2x^2 - 10x - 11 = 0$
32. $7x^2 - 5x + 2 = 0$

Find k in each of the following:

33. $3x^2 - 12x + k = 0$, when 2 is a root.
34. $x^2 + 2(k + 2) + 9k = 0$ when the left-hand member is a perfect square.
35. $x^2 - 15 - k(2x - 8) = 0$ if the roots are real and equal.

Solve the simultaneous equations:

36. $2x + y = 8 \qquad x^2 - 2y^2 = 17$

*37. $2x + y = 2 \qquad 4x^2 + 6xy + 2x - 6y + 1 = 0$

38. $4x+7=y^2 \qquad 2x=4-y$

39. $x+2y-5=0 \qquad x^2-x-4y-10=0$

40. $2x-3y=5 \qquad 4x^2+4y^2=25$

41. $5x+2y=4 \qquad x^2-3x=y+1$

42. $x^2-y^2=9 \qquad x+2y-3=0$

43. $2x+y=10 \qquad xy=8$

*44. $\dfrac{x}{2y}-\dfrac{y}{3x}=\dfrac{29}{24} \qquad 4x-y+2=0$

*45. $x^2+5xy+3y^2=3 \qquad 3x^2+7xy+4y^2=5$

Test on Chapter XI

1. Expand to four terms $(a-b)^n$.

Expand:

2. $(a+b)^5$ 3. $(x-y)^5$ 4. $(2x+5)^6$

5. $\left(2a^2+\dfrac{1}{a}\right)^4$ 6. $(a^2-4)^4$ 7. $\left(x-\dfrac{2}{y}\right)^7$

Write the first four terms of the expansion of:

8. $\left(x^2-\dfrac{2}{x}\right)^{12}$ 9. $\left(x^3-\dfrac{1}{x^2}\right)^8$ 10. $(2x-5y)^{15}$

11. $\left(3m-\dfrac{1}{3n}\right)^{14}$ 12. $(a+b)^{200}$

Without the use of logarithms find the amount (A) to which a given sum (S) will grow if compounded at a rate r in each example below. Answer to the nearest cent. $A=S(1+r)^n$

13. \$200 at 4% for 5 years compounded annually.

*14. \$500 at 4% for 3 years compounded semi-annually.

*15. \$325 at 5% for 8 years compounded semi-annually.

Find the terms indicated; simplify those that are not starred:

16. $(x + 2y)^{10}$ seventh term

17. $\left(2x - \frac{y}{4}\right)^8$ fifth term

18. $\left(3a - \frac{2b^2}{3}\right)^7$ fourth term

19. $(\sqrt{2} - a\sqrt{3x})^{10}$ seventh term

*20. $(a - b)^{40}$ fourteenth term

21. $\left(x^3 - \frac{1}{x^2}\right)^8$ middle term

*22. $\left(4 + \frac{3}{x}\right)^{25}$ sixth term

*23. $(x^2 + x)^{11}$ the term containing x^{15}

24. $\left(2x - \frac{3}{x}\right)^9$ the term without x

*25. In the expansion $\left(x - \frac{4}{x}\right)^{10}$ find the coefficient of the term containing x to the first power.

Tests Similar to Those Set for Admission to College

College Entrance Test I

1. (*a*) Factor $\frac{y^2}{x^2} + \frac{x^2}{y^2} - 2$

(*b*) Simplify $\frac{5a}{x-3} - \frac{a}{3-x}$

2. Given the formulas $F = \frac{mv^2}{r}$ and $r = \frac{\frac{1}{2}vt}{\pi}$, solve the second for v and substitute this value for v in the first. Find F when $m = 80$, $r = 2\frac{1}{2}$, $t = 10$, and $\pi = \frac{22}{7}$.

3. Solve and check:

$$\frac{x}{2}+\frac{y}{2}=\frac{7}{4} \qquad \frac{2x+y}{3}=\frac{1}{2}\left(4x-\frac{y}{3}\right)+\frac{5}{6}$$

4. (*a*) With the help of the formulas insert 5 arithmetic means between 1 and 10. Write out the series and show that you are correct.

(*b*) Find and simplify the middle term of the expansion of $\left(x-\frac{1}{x}\right)^{10}$

(*c*) Find by formula the sum of 6 terms of the series $2-3+\frac{9}{2}-\ldots$

5. (*a*) Simplify, and check by setting $a=2$, $x=3$, $b=4$:

$$\frac{2a-b}{x-2a}-\frac{b-4a}{2a+x}-\frac{3x(b-2a)}{4a^2-x^2}$$

(*b*) If $x=2$, find the value of each of the following expressions:

$$x^{\frac{2}{3}} \qquad \frac{(2x)^{\frac{3}{2}}}{8} \qquad \frac{1}{(x^{-1})^{\frac{4}{3}}} \qquad x^{-3} \qquad (x^{-\frac{2}{3}})^6$$

6. The price of a diamond varies as the square of its weight. If a diamond weighing 12 grains costs \$360, what will an 18-grain diamond cost?

7. One leg of a right triangle is 156.0 ft., and the adjacent acute angle is 37° 40′. Find the other leg.

8. A man invested \$11,700 in two enterprises. He found that at the end of the first year he had gained 6% on one of them and lost 4% on the other. His net profit for the year was \$282. How much had he invested in each enterprise?

9. What will the graph of each of the following equations be?

$x+y=3 \qquad x^2+2y^2=81$ Plot the graphs on

the same axes and thus determine what values for x and y satisfy both equations.

***10.** A boy starts into the country on his bicycle and travels 12 miles when his chain breaks. He spends 15 minutes trying to repair it, and then starts to walk back at a speed 3 miles an hour less than that at which he rode out. After 40 minutes he is given a lift by a passing automobilist and carried the remaining distance at a speed 14 miles an hour greater than that at which he rode. He reaches home 35 minutes earlier than he would have had he made the entire journey on his bicycle. Find the rate at which he rode his bicycle.

College Entrance Test II

1. (*a*) Factor $x^4 - (5x+6)^2$

(*b*) Simplify $4\sqrt{\frac{2}{3}} - \frac{1}{6^{-\frac{1}{2}}}$

*(*c*) Simplify $a^{\frac{1}{3}}(a^{-\frac{2}{3}} - a^{\frac{5}{3}})^{-1}$

2. Solve for x, y, and z:

$$x + y + 2z = 1 \qquad 2z + 3y = 4 - 2x \qquad 4x + 9y = 16 - 2z$$

3. From the top of a rock rising to a height of 24.06 ft. above the edge of a river, the angle of depression of a point on the other side is 32° 43′. Find the width of the river.

4. A man walked six miles into the country. When he was ready to return, he discovered that he had at his disposal one half hour less than the time he had taken going out. He walked back at a speed two miles an hour greater, and arrived on time. Find both rates.

5. Solve for x and y: $\quad x^2 + xy = 24 \qquad 3x = y + 4$

6. (*a*) How many terms of the series 4, 12, 36... must be taken to give a sum of 484?

(*b*) Find the sum of the first 150 consecutive multiples of six.

7. Solve graphically, estimating roots as closely as possible:

$$y + 2x = 10 \qquad xy = 8$$

8. (*a*) Solve, leave answers in simplest radical form, and check by use of the relation of roots to coefficients:

$$x^2 - \frac{5x}{3} - \frac{4}{3} = 0$$

(*b*) Expand to five terms $(x + y)^n$

9. A man bought two farms for $2800 each. The larger contained ten acres more than the smaller, but he paid $5 more an acre for the smaller. How many acres did he buy? How many acres were there in each farm?

10. A rectangular court can be paved with 288 square tiles. If a tile 3 in. greater on a side is used, only 162 tiles will be needed. Find the size of each tile.

College Entrance Test III

1. (*a*) Factor $3x^2 - 17x + 24$

(*b*) Simplify $\dfrac{a + b}{a^{-1} + b^{-1}}$

(*c*) Factor $m^2 - 4p^2 + 12p - 9$

(*d*) Simplify $\sqrt{180} - 2\sqrt{5} + 15\sqrt{\frac{4}{5}}$

2. Solve for x, y, and z:

$$x + y - z = 3 \qquad -3x + y = -8 \qquad -x + z = 4$$

3. Simplify $\dfrac{m^2 + m - 13}{12 + m - m^2} - \dfrac{m + 2}{m + 3} + \dfrac{m - 3}{m - 4}$

4. (*a*) Which term of the series $2\frac{1}{2}$, $3\frac{3}{4}$, $5 \ldots$ is 45? Find the sum of seven terms.

(*b*) The numbers $2x-4$, $5x-7$, $10x+4$ form a geometric progression. Find x.

*5. Expand $(\sqrt{2a}-\sqrt{x})^5$.

6. (*a*) Solve for x and check $\sqrt{3x+7}-x=3$.

(*b*) The equation $2x^2-2x+k=0$ has equal roots. Find k.

7. A broker bought some stock for \$1000. The price advanced \$20 a share, and he sold all but one share, receiving \$1080. Find how many shares he bought.

8. Solve graphically:

$$y=2x^2-5x-3 \text{ and } x+y=3$$

9. Seen from the top of a vertical cliff 125 ft. 6 in. high, the angle of depression of a ship at sea is 12.45°. How far out is the ship?

10. A chauffeur engages to accomplish a journey of 100 miles in a specified time. After traveling 50 miles at an average speed that will enable him to keep his engagement, he is delayed 20 minutes. He then completes his journey at a speed 5 miles an hour greater than that at which he set out and reaches his destination on time. Find his original speed.

College Entrance Test IV

1. (*a*) Factor $4z^4-25z^2+36$.

(*b*) Factor $27+63x-24x^2$

(*c*) Simplify $\dfrac{m}{2m+1}+\dfrac{m^2+m+2}{2m^2-5m-3}+\dfrac{m-1}{3-m}$

2. (*a*) Simplify $\dfrac{1}{3}\left(\dfrac{3}{4}\right)^{\frac{1}{2}}+\dfrac{1}{2}\sqrt{\dfrac{1}{3}}-\dfrac{5}{75^{-\frac{1}{2}}}$

(*b*) Simplify $\dfrac{b^{\frac{1}{2}}}{b^{-\frac{1}{2}}+b^{-\frac{3}{2}}}$

3. Solve and check: $\frac{2x}{3} - \frac{3y}{4} = 10 \qquad 0.6x + 0.7y = -2.$

4. Solve, pairing answers correctly:

$$3x + y = 1 \quad 3x^2 - 4xy + y^2 = 7$$

5. (*a*) Obtain by formula the 10th term and the sum of the first 10 terms of the progression $2\frac{1}{3}, 4, 5\frac{2}{3}\ldots$

(*b*) Find the value of \$500 after 5 years at 4% compounded quarterly.

(*c*) Derive the formula for the sum of the first n terms of a geometric progression.

6. (*a*) Solve, leave roots in simplest radical form, and check by use of relation of roots to coefficients:

$$4x^2 - 10x + 3 = 0$$

(*b*) Write and simplify the first three terms of the expansion of:

$$\left(\frac{1}{x} - \frac{\sqrt{x}}{y}\right)^{12}$$

(*c*) Write and simplify the 9th term of the expansion of $(6b - 3a)^{11}$.

7. A mountain peak 2143 ft. high casts its shadow at the threshold of a farmhouse on the plain below when the sun's rays make an angle $12° 37'$ with the horizontal. Find the distance of the farmhouse from the mountain. (That is, the horizontal distance from the house to a point perpendicularly beneath the peak.)

8. Solve graphically $x^2 - y^2 = 9$ and $x + 2y - 3 = 0$.

9. Two automobiles leave the same place and travel over the same route without stopping. The first travels 12 miles the first hour, $12\frac{1}{2}$ miles the second, 13 the third, etc. The second automobile leaves 4 hr. 40 min. after the first and travels at the uniform rate of 33 miles an hour. How long after the first

starts will the two cars be together? How far will each have traveled?

10. A farmer has enough wire netting to build a rectangular inclosure with length 2 ft. more than 7 times the width. He finds, however, that if he builds a square inclosure, using the same amount of netting, the area of the inclosure will be increased 361 sq. ft. How many feet of netting has he?

College Entrance Test V

Part I. No credit will be given for answers partially correct.

1. Factor $63 + 24\,m - 15\,m^2$
2. Factor $(a - b)^2 - 3(a - b)$
3. Factor $n^3 - m^2n$
4. Solve $x + y = 8 \quad x - y = 2$
5. Simplify $\dfrac{3\,m}{m - n} + \dfrac{3\,y}{n - m}$
6. Simplify $\dfrac{2}{x + 2} - \dfrac{1}{x}$
7. Simplify $\sqrt{\frac{5}{3}} - \sqrt{\frac{3}{5}}$
8. If $x = 2$, find the value of $\dfrac{2}{3}(x + 4) - \left(\dfrac{5\,x}{3} - 1\right)$
9. If $x = 2$, find the value of $\dfrac{\sqrt{(2\,x)^3}}{8}$
10. Solve $\dfrac{2\,x - 1}{2} = \dfrac{x + 3}{5}$
11. Solve $x^2 - 10\,x - 39 = 0$
12. Solve $3\,x^2 + x - 1 = 0$. Leave answers in radical form.
13. State the sum and product of the roots of the equation $x^2 - 2\,x - 15 = 0$. Do not solve.
14. Simplify: $\dfrac{5\,y^{-2}}{y^{-1} - y^{-2}} - y^3$
15. Write and simplify the first three terms of the expansion of $(3\,x - y)^7$.

16. The cost of the plate for printing a map is \$50, and the cost for printing is 5 cents a copy. Write a formula which will give the total cost, T, for n copies.

17. Find by formula the sum of 50 terms of the series

$$1 + 3 + 5 + \ldots$$

18. Find the fractional equivalent of 0.13131 . . .

19. One leg of a right triangle is 12.0 inches, and one of the angles 50.0°. Find the other leg.

20. Use logarithms to find the cube root of 517.

Part II

1. Solve for x and y, pairing your answers:

$$2x - y = 5 \quad x^2 - 4y^2 = 5$$

2. A and B can do a piece of work in 12 days. After the work is half done, B stops work and A finishes in 15 days more. How much of the work can each do in one day?

3. One side of a given rectangle is 2 in. less than a certain standard, x, and the other side 3 in. greater than this standard. Write a formula for the area, y, of this given rectangle and plot the graph of y for values from $x = 1$ to and including $x = 6$. Give the approximate value of y when $x = 4\frac{1}{2}$ in.

4. The hypotenuse of a right triangle is 241.5″, and one leg is 95.28″. Find the angles.

5. Twenty-five hundred dollars invested at simple interest amounted to \$2800 in a given time. The same amount invested at a rate 1% higher and allowed to stand two years longer amounted to \$3200. Find the first rate.

6. A man drove from one town to another in 14 hours. Returning by a route ten miles longer, he traveled at a speed which enabled him to cover each 20 miles in half an hour less time than

going out. He reached home in the same number of hours that he took to go out. How far apart are the towns?

College Entrance Test VI

Part I. No credit will be given for answers partially correct.

1. If $a = 4$, $b = 3$, $c = 2$, $d = 2$, what is the value of $\frac{a}{b} + \frac{(b - d)}{dc}$?

2. Solve $\frac{2}{3}x + 17 = \frac{5}{6}x - 19$.

3. A book cost c cents to make and e cents to sell. The profit is p cents. What is the selling price of seven copies?

4. $C = \frac{5}{9}(F - 32)$. Find C when $F = 113$.

5. Multiply $(3 - x)(x + 3)$. **6.** Factor $3 - m - 10m^2$.

7. Simplify $8^{-\frac{2}{3}} + 6^0 - 4^{\frac{3}{2}}$. **8.** Solve for r: $V = \frac{1}{3}\pi r^2 h$

9. Write without negative exponents $\frac{8^{-3}m^2n^{-2}}{2^{-2}m^{-1}n^2}$

10. Write the sum and product of the roots of the equation $4x + 5 = 3x^2$.

11. Write the fifth term of the expansion of $(a + b)^7$.

12. Simplify $\left(\frac{a^2}{b^2} - 1\right) \div a\left(\frac{a}{b} - 1\right)$.

13. Simplify $\frac{2a^{-1} + 1}{a^{-1}}$.

14. Solve $\sqrt{2x - 14} - 4 = 0$.

15. In an Arithmetic Progression $s = 8$, $a = -1$, $l = 5$. Find n.

16. Solve for b: $A = \frac{2}{3}ab^2$.

17. Find by logarithms the value of $\sqrt{\dfrac{15.0}{2 \cdot 4.57}}$

18. Factor $2a^2 - 8$.

19. Solve, leaving answers in radical form $2x^2 + 5x + 1 = 0$.

20. Find by formula the sixth term of $\frac{2}{3} + 2 + 6 + \ldots$

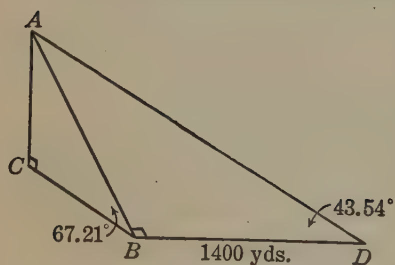
Part II

1. Solve, leave answers in simplest radical form, and check:

$$-x^2 + y^2 = 16 \quad y = 2x + 3$$

2. Write an equation which has for its roots -2 and $+5$. Set this equation equal to y, and plot its graph. From this graph find x when $y = 8$.

3. The accompanying figure, drawn in perspective, gives the measurements needed to determine the height of an entirely inaccessible mountain peak. Angle $C = 90°$, angle $ABD = 90°$. Find the height AC.



4. A goldsmith has two alloys, one $\frac{3}{4}$ pure gold, the other $\frac{5}{12}$ pure gold. How many ounces of each shall he take to get 10 oz. which shall be $\frac{2}{3}$ pure gold?

5. A stranger changing at a certain town A for town B learns that there is a bus in one hour and a quarter. Rather than wait, he starts out on foot for B, traveling at 4 miles an hour. When the bus overtakes him, he boards it, is carried the remaining distance from A to B at 24 miles an hour and reaches his destination one hour 40 minutes after being picked up. Find the distance from A to B.

A New York Regents' Examination in Intermediate Algebra

Part I. Answer all questions in this part. Each question has $2\frac{1}{2}$ credits assigned to it; no partial credit should be allowed. Each answer must be reduced to its simplest form.

1. Simplify $\sqrt{1-(\frac{1}{2})^2}$. Leave answer in radical form.

2. If $v = \dfrac{1}{c} - 1$ find the value of $(v+1)^2$ in terms of c.

3. Given $\log x = 0.3156$; find $\log \dfrac{10}{x}$.

4. Simplify $(.125)^{\frac{1}{3}} + (16)^{-\frac{3}{4}}$.

5. In the arithmetic progression $\frac{3}{4}, \frac{1}{2}, \ldots$ find the 12th term.

6. Solve for n the formula $I = \dfrac{nE}{R + nr}$.

7. What is the binomial factor of $x^3 - 2x^2 - x - 6$?

8. The area of a rectangle is 240 and its dimensions have the ratio 5:3; find the longer dimension.

9. In the equation $3x^2 - 16x = 8$, what is the product of the roots?

10. The sum of the roots of a quadratic equation is 8 and their difference is 4; write the equation.

11. Solve for x: $\sqrt{x^2 + 27} = 9 - x$.

12. Considering the first five powers of 2 as a progression, write the formula that would be used in finding their sum.

13. Simplify $3x^0 - (3x)^0 - 1^{3x}$ if $x = 2$.

14. Factor $x^{2a} - 3x^a - 10$.

15. Given $(x+y):(x-y) = 5:2$, find $x:y$.

16. What is the value of $x^2 - 2x - 3$ when $x = \sqrt{2} - 1$?

17. If $y = \frac{-12}{x}$ and x is positive, does y increase or decrease as x increases?

18. Will the graphs of $3x - y = 1$ and $6x - 3 = 2y$ have a point in common? (Answer yes or no.)

19. What is the value of y for the intersection of $4x - 3y = 12$ with the y-axis?

20. If $2x - 3$ is a factor of $kx^2 - 7x - 3$, $k = ?$

Part II. Answer five questions in this part. Full credit will not be granted unless all operations (except mental ones) necessary to find results are given; simply indicating the operations is not sufficient. Each answer should be reduced to its simplest form. In the examination in intermediate algebra the use of slide rule will be allowed for checking, provided all computations with tables are shown on the answer paper.

1. The perimeter of a rectangle is 20 inches. If the length is increased by 2 in. and the width is decreased by 1 in., then the area is 24 sq. in. What are the dimensions of the original rectangle?

2. Brown and Smith own two tracts of land, Brown's tract being $\frac{2}{3}$ as large as Smith's. After selling Smith 60 acres, Brown owns $\frac{3}{7}$ as much as Smith. How many acres did Smith own at first?

3. Find three numbers in geometric progression whose sum is 26 and whose product is 216.

4. In the formula $K : K' = a^2 : a'^2$, $K = 1600$, $K' = 7290$ and $a' = 310.6$. By the use of logarithms find the value of a.

5. (*a*) One root of the equation $2x^2 - 15x + k = 0$ is twice the other; what is the value of k?

(*b*) Form the equation whose roots are $1 + \sqrt{3}$ and $1 - \sqrt{3}$.

6. Solve the following set of equations, correctly group your answers, and check one answer:

$$8/x - 9/y = -1 \qquad x - 3y = -5$$

7. State whether *each* of the following statements is true or false: (Write the letters a, b, c, d, e in a column and then write the word *true* or *false* after each letter.)

(a) The logarithm of 2^0 is zero.

(b) An arithmetic progression becomes a geometric progression when each of its terms is multiplied by 2.

(c) $2^a \times 2^a = 4^{2a}$.

(d) If the length of a rectangle is decreased by d feet and its width is increased by 3 feet, its area is not changed.

(e) The roots of equation $2x^2 - x = 1$ are rational.

8. (a) Form a table of values of $y = x - x^2$ by giving x all integral values from -2 to $+3$ inclusive.

(b) Draw the graph of the equation.

*(c) By means of the graph drawn in answer to b, determine the nature of the roots of $x - x^2 = 3$ and $x - x^2 = -4$.

A New York Regents' Examination in Intermediate Algebra

Part I. Answer all questions in this part. Each question has $2\frac{1}{2}$ credits assigned to it; no partial credit should be allowed. Each answer must be reduced to its simplest form.

1. By means of the formula $K = \frac{b}{2}\sqrt{\left(a - \frac{b}{2}\right)\left(a + \frac{b}{2}\right)}$ find K when $a = 20$ and $b = 32$.

2. Find the value of x^{-1} if $2x^{\frac{1}{2}} = 1$.

3. Simplify $\left(\frac{b}{a-b} + 1\right) \div \frac{a}{b-a}$

4. Factor $3x^3 - 5x^2 - 2x$.

5. Find the roots of $2x^2 + x = 6$.

6. Find the root common to the equations $x^2 = 2x + 3$ and $x^2 = 9$.

7. Solve the following set of equations for z:

$$3x - 2y - 5z = 11 \qquad y = 2x \qquad x = 4$$

8. Write in the form $x^2 + px + q = 0$ the quadratic equation whose roots are $2 + \sqrt{3}$ and $2 - \sqrt{3}$.

9. Find the product of the roots of the equation

$$x^2 + 3x - 2 = 0.$$

10. If the discriminant of a quadratic equation is 64, are the roots rational or irrational?

11. Does $\sqrt{\dfrac{x}{y}}$ equal $\dfrac{x^{-\frac{1}{2}}}{y^{-\frac{1}{2}}}$?

12. Solve for r the formula $S = \dfrac{lr - a}{r - 1}$.

13. Write the formula that would be used in finding the sum of the first 50 odd positive integers.

14. Solve for x the equation $\sqrt{\dfrac{1}{x}} = 2$.

15. If $y = \dfrac{10 - x}{x}$ does y increase or decrease as x increases from 1 to 5?

16. Does $\dfrac{\log x}{\log y}$ equal $\log x - \log y$?

17. Given $\log x^2 = 0.6290$, find $\log 10x$.

18. What integral multiple of $\log 2$ is $\log 32$?

19. What must be the value of a if the graph $3x - 5y = a$ passes through the point where $x = 3$ and $y = 1$?

20. How many more articles can I buy for d dollars when the price of each is 20 cents than when the price of each is 25 cents?

Part II. (See comments under preceding Regents' examination, Part II.)

1. (*a*) Factor $x^3 - 7x - 6$

(*b*) Simplify $(x+1)\sqrt[3]{16x^2} + 4x\sqrt[3]{\frac{1}{4}x}$

2. Find by the use of logarithms the cube root of 18.43 to three decimal places.

3. The speed of an airplane is 90 miles an hour in calm weather. Flying with the wind, it can cover a certain distance in 4 hours but, when flying against the wind, it can cover only $\frac{3}{5}$ of this distance in the same time. What is the velocity of the wind?

4. Two adjoining lots of different size, each of which is a square, front on the same street. The combined area is 25 sq. rods. The perimeter of the two lots considered as a single lot is 22 rods. Find the dimensions of each lot.

5. In a geometric progression the second term exceeds the first term by 4 and the sum of the second and third terms is 24; find the sum of the first five terms of *one* of the two possible progressions.

6. Solve the following set of equations, group the answers, and check *one* set of roots:

$$x^2 + 6xy = 28 \quad xy + 8y^2 = 4$$

7. The diagonal of a rectangle is 5 and the perimeter is 14.

(*a*) Letting x and y represent the dimensions, write two equations in x and y expressing the above facts.

(*b*) Draw the graphs of the two equations written in answer to (*a*).

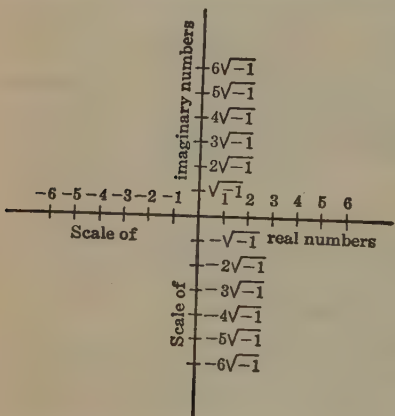
(*c*) From these graphs determine the dimensions of the rectangle.

CHAPTER XIII

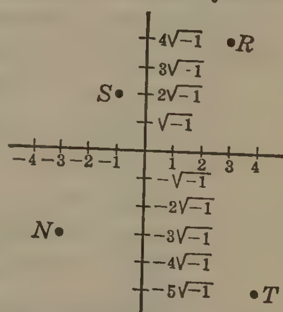
*ADDITIONAL MATERIAL FOR STUDY

Imaginary Numbers

Graphs. Every real number may be represented, as you know, by a point on an indefinite line, and every point on this line can represent some real number, either an integer, a fraction, or an irrational number. In other words, after a zero point and a unit on a scale have been selected, every real number may be represented by a distance and a direction.



On another scale perpendicular to the first, there can be represented every pure imaginary number. This graphical representation adapts itself to the theory of imaginary numbers just as the ordinary scales adapt themselves to real numbers; it is a useful device not only to picture the facts about imaginary numbers, but also to aid in the study of electricity and other subjects.



1. The accompanying illustration shows how complex† numbers can be represented graphically. The point R represents $3 + 4\sqrt{-1}$; S represents $-1 + 2\sqrt{-1}$. If $i = \sqrt{-1}$, the point T represents $4 - 5\,i$. What

† A complex number is part imaginary and part real.

complex number is represented by N? Plot in the same way: $3-4\sqrt{-1}$, $2+\sqrt{-1}$, $3-\sqrt{-1}$, $2-3i$, $-3+2i$, $-i$, $-1-i$.

Show that the multiplication of any number by -1 may be represented graphically by a change of 180° in direction. It is interesting to notice that this idea fits into our scheme for representing imaginary numbers. Multiplying twice by $\sqrt{-1}$ has the same effect on a number as multiplying once by -1; therefore, multiplication by $\sqrt{-1}$ may be thought of as causing a 90° change of direction. Illustrate by pointing out on the graph the following points: 4, $4\sqrt{-1}$, $4\sqrt{-1}\sqrt{-1} \equiv -4$ $-4\sqrt{-1}$, $-4\sqrt{-1}\sqrt{-1} \equiv -4(-1) \equiv 4$.

Operating with imaginary numbers.

1. $2\sqrt{-3}+5\sqrt{-3}=7\sqrt{-3}$ and $5i+6i=11i$

Unite: $4\sqrt{-1}+7\sqrt{-1}-8\sqrt{-1}$; $3i+5i-10i$;
$3\sqrt{-27}-2\sqrt{-75}-\sqrt{-\frac{1}{3}}$ (Suggestion: $\sqrt{-27}=3\sqrt{-3}$.)
$1+i-(5+3i)$; $4-2i-(4-2i)$; $3.6+5.8i-3.2+2.7i$

2. $\sqrt{-4}=2\sqrt{-1}\equiv 2i$ $\sqrt{-3}=\sqrt{3}\cdot\sqrt{-1}\equiv\sqrt{3}\,i$

3. Express in terms of i: $\sqrt{-25}$ $\sqrt{-27}$ $\sqrt{-5}$
$\sqrt{-81}$ $\sqrt{-72}$ $\sqrt{-\frac{1}{4}}$ $\sqrt{-\frac{4}{3}}$

4. Powers of *i*. $i^2=-1$ because $\sqrt{-1}\times\sqrt{-1}\equiv -1$ by definition of square root. $i^3=-i$ or $-\sqrt{-1}$ because $i^3\equiv i^2\cdot i\equiv -\sqrt{-1}$.
$i^4=1$ because $i^4\equiv i^3\cdot i\equiv -\sqrt{-1}\cdot\sqrt{-1}$ Find values of i^5, i^6, i^7, i^8 and show that they repeat the values found above.
$i^{17}=i^{16}\cdot i\equiv (i^4)^4\cdot i\equiv i$. In this way find values of i^{36}; i^{37}; $2i\cdot 3i$.

5. $\sqrt{-3}\times\sqrt{-3}=-3$ by definition of square root.
$\sqrt{-3}\cdot\sqrt{-5}=\sqrt{3}\,i\cdot\sqrt{5}\,i\equiv\sqrt{15}\,i^2\equiv-\sqrt{15}$. This result is

somewhat surprising, and must be carefully considered before attempting to multiply imaginary numbers.

6. Divide $\sqrt{-6}$ by $\sqrt{-2}$. *Plan of work:* $\dfrac{\sqrt{6}\,i}{\sqrt{2}\,i} \equiv \dfrac{\sqrt{6}}{\sqrt{2}} \equiv ?$

7. Divide $\sqrt{-4}$ by $\sqrt{-2}$; 1 by $\sqrt{-1}$.

8. Divide 1 by $(\sqrt{-1})^5$.

Plan: $\dfrac{1}{(\sqrt{-1})^5} \equiv \dfrac{1}{i^5} \equiv \dfrac{i^3}{i^3} \cdot \dfrac{1}{i^5} \equiv \dfrac{i^3}{i^8} \equiv ?$ But $i^8 \equiv (i^4)^2 \equiv 1$.

9. Divide 1 by i^3; $2\,i$ by i^2.

10. Add: $4+3\,i \qquad 3+2\,i \qquad 5+8\,i$

11. Collect: $3\,i+2-6\,i-7+8\,i+4$

12. Multiply $2+3\,i$ by $1+2\,i$.

Solution:

$$\begin{array}{r} 2+3\,i \\ 1+2\,i \\ \hline 2+3\,i \\ +4\,i+6\,i^2 \\ \hline 2+7\,i+6\,i^2 \end{array}$$

But $i^2 = ?$ Therefore the product is. . . .

13. Multiply $(\sqrt{3}+2\,i)(\sqrt{2}+3\,i) \qquad (a+bi)(a-bi)$

$(a\sqrt{b}+c\sqrt{d}\,i)(a\sqrt{b}-c\sqrt{d}\,i)$

14. Divide $1+\sqrt{3}\,i$ by $1-\sqrt{3}\,i$.

Solution: $\dfrac{1+\sqrt{3}\,i}{1-\sqrt{3}\,i} = \dfrac{1+\sqrt{3}\,i}{1-\sqrt{3}\,i} \cdot \dfrac{1+\sqrt{3}\,i}{1+\sqrt{3}\,i} \equiv \dfrac{1+2\sqrt{3}\,i+3(i)^2}{1-3(i)^2}$

$$\equiv \frac{-2+2\sqrt{3}\,i}{1+3} \equiv ?$$

15. Divide: $\frac{3}{\sqrt{2}+i}$ $\frac{1-i}{1+i}$ $\frac{1+2i}{1-i}$. Check by division the product in Example 12.

16. Solve the quadratic equation $x^2 + x + 2 = 0$, express the roots in the "i form" and check them by means of the relation between the roots and the coefficients in the equation.

17. Find the three cube roots of 1.

Solution: Let $x^3 = 1$, then $x^3 - 1 = 0$ or $(x-1)(\ ?\ ) = 0$.

$$x = 1, \quad x = \frac{-1+\sqrt{3}\,i}{2}, \quad x = \frac{-1-\sqrt{3}\,i}{2}. \quad \text{Explain.}$$

18. Find the three cube roots of -8.

19. Write the quadratic equation of which the roots are $2 \pm 3\,i$. Try to write the equation which has roots 5 and $2 - 6\,i$. Tell why imaginary roots appear in pairs, if the coefficients of the equation are real. The roots of the equation

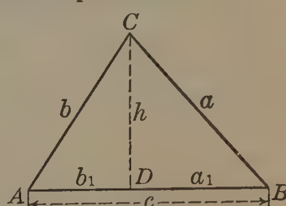
$$x^4 - 4\,x^3 - 8\,x + 32 = 0 \text{ are } 2,\ 4,\ -1+3\,i,\ -1-3\,i,$$

the last two forming a "conjugate pair." The roots of the equation $x^4 - x^2 + 1 = 0$ are $\frac{1}{2}(\sqrt{3} \pm i)$, $\frac{1}{2}(-\sqrt{3} \pm i)$; that is, they consist of two conjugate pairs.

Trigonometry

The introduction of two or three additional formulas will enable you to solve many new trigonometric problems.

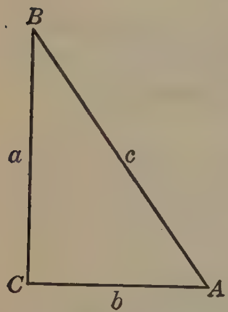
1. The Law of Sines. In the oblique triangle illustrated, CD is perpendicular to AB; that is, $h \perp c$. What is the sine of A? the cosine of A? $\sin B$? $\cos B$? Find $\frac{\sin A}{\sin B}$.



Solution: $\sin A = \frac{h}{b}$ $\quad \sin B = \frac{h}{a}$ $\quad \frac{\sin A}{\sin B} = \frac{\frac{h}{b}}{\frac{h}{a}} = ?$

This is called the "Law of Sines." It may also be written:

$$\frac{\sin B}{\sin C} = \frac{b}{c} \quad \text{and} \quad \frac{\sin A}{\sin C} = \frac{a}{c}$$



2. Find the value of $\sin^2 A + \cos^2 A$ where A is any acute angle.

Plan of work: Draw ABC, a right triangle of reference. $\sin^2 A = \frac{a^2}{c^2}$ and $\cos^2 A = \frac{b^2}{c^2}$. $\sin^2 A + \cos^2 A = \frac{a^2 + b^2}{c^2}$. But $a^2 + b^2 = c^2$ (why?). Therefore, $\sin^2 A + \cos^2 A = 1$.

3. **The Law of Cosines.** In the figure for Example 1, page 339, ① $a^2 = h^2 + a_1^2$. But ② $h = b \sin A$ and ③ $a_1 = c - b_1 \equiv c - b \cos A$.

Substituting ② and ③ in 1: gives ④ $a^2 = b^2 \sin^2 A + c^2 - 2bc \cos A + b^2 \cos^2 A = b^2(\sin^2 A + \cos^2 A) + c^2 - 2bc \cos A$.

But ⑤ $\sin^2 A + \cos^2 A = 1$ (see preceding example), and therefore, ⑥ $a^2 = b^2 + c^2 - 2\,bc \cos A$.

This is called the "Law of Cosines" and may also be written:

$$c^2 = a^2 + b^2 - 2\,ab \cos C, \text{ or } b^2 = a^2 + c^2 - 2\,ac \cos B.$$

The sines and cosines of obtuse angles are not given in the tables of trigonometric functions. This difficulty is overcome by the use of two facts: (1) the sine of an obtuse angle equals the sine of its supplement, and (2) the cosine of an obtuse angle equals the cosine of its supplement. (These equalities are numerical; the sine of 120° = the sine of 60°, but the cosine of 120° = *minus* the cosine of 60°.)

Applications. In the law of sines and the law of cosines, which can be proved to hold for *all* triangles, we have two powerful means for solving triangles. The first one enables us to solve

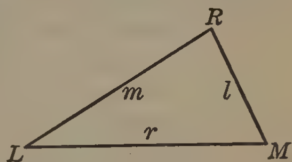
triangles when we know a side and two angles or two sides and an angle opposite one of them; the second enables us to solve triangles when we know two sides and the included angle or three sides.

Find the sides or angles called for:

4. $A = 11° 18'$, $B = 47° 12'$, $a = 567.7$ ft. Find b.

5. $A = 80.00°$, $B = 55.25°$, $c = 1000$ ft. Find a. (You will need angle C. How can this angle be found? $A + B + C =$ how many degrees?)

6. To find the distance from the rock (R) to a lighthouse (L), the line LM and the angles at L and M are measured and found to be, $r = 1500$ ft., $L = 24.25°$, and $M = 60.38°$.

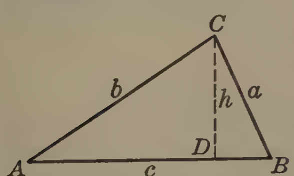


Use logarithms to find m.

7. $a = 45.28$ ft., $b = 71.84$ ft., $B = 52° 21'$. Find A.

8. In a certain triangle $b = 10$ ft., $c = 20$ ft., and $A = 45°$. Find a.

9. Find the angles of a triangle with sides $a = 12$, $b = 15$, $c = 18$.



10. Find the area of triangle ABC shown in the figure.

Plan of solution: Area $= \frac{1}{2}hc$. But $h = a \sin B$; therefore, area, $K = \frac{1}{2}ac \sin B$. Show that $K = \frac{1}{2}ab \sin C$, $\frac{1}{2}bc \sin$?

11. Find the area of the triangle of Example 8.

12. Find the area of the triangle of Example 9.

The three formulas developed in this section serve chiefly to give some simple illustrations of the methods to which algebra

is leading in future courses. They are practical, as you can readily see, and acquaintance with them is by no means beyond your present powers. These formulas can also be used to simplify investigations in geometry.

Simultaneous Quadratic Equations

1. Solve and check: ① $x^2 + y^2 = 17$ ② $4x + y = 15$.

Whenever one of a pair of equations is quadratic and the other linear, it is possible to solve them by the method of substitution. In the example above, ② can be solved for y and the result substituted in ①. It is sometimes needful, however, to solve a pair of equations which are both quadratic. Two or three pairs of this kind are touched upon on pages 290, 291. The method of substitution cannot as a rule be used in such a pair. Certain "ingenious devices" must be resorted to. Because each pair of quadratic equations must in general be solved by a method peculiar to itself, a lengthy discussion of the subject would hardly be profitable in this course. The methods of 290, 291 can, however, be carried somewhat farther. The following will serve as illustrations:

2. Solve ① $xy = -10$ ② $x^2 + y^2 = 29$. Try this example by the method of example 2, page 291, then complete the following:

Plan of solution: ③ $2xy = -20$ ① × 2

④ $x^2 + 2xy + y^2 = 9$ ② + ③

⑤ $x + y = \pm 3$ Square root of ④.

Solve the equations of ⑤ with ① simultaneously by the method of substitution, and check your answers. You will notice that the method of solution outlined above gives somewhat easier substitution.

3. Solve $x^2 + y^2 = 106$ $xy = 45$.

4. Solve $x^2 + y^2 = 101$ $xy = 10$.

5. Solve ① $x^2 + xy = 60$ ② $xy + y^2 = 84.$

Plan of solution:

③ $x^2 + 2xy + y^2 = 144$ (How was this equation obtained from ① and ②?)

④ $x + y = \pm 12$ Square root of ③.

Complete the solution and check the roots.

6. Solve: ① $x^2 + y^2 = 10$ ② $x^2 - xy + y^2 = 7.$ (These two equations are *symmetric*; that is, their value is unchanged if x is substituted for y and y for x. Pairs of equations of this type, containing the xy term, can usually be solved by eliminating the constant terms.)

Plan of solution: ③ $7x^2 + 7y^2 = 70$ ① × 7

④ $10x^2 - 10xy + 10y^2 = 70$ ② × 10

⑤ $3x^2 - 10xy + 3y^2 = 0$ ④ − ③

⑥ $(3x - y)(x - 3y) = 0$

Solve equations ⑥ with ① as a pair, and check. Compare this example with Example 4 on page 292.

***7.** Solve: ① $\frac{1}{x^3} - \frac{1}{y^3} = 91$ ② $\frac{1}{x} - \frac{1}{y} = 1$

Plan of solution: ③ $\frac{1}{x^2} + \frac{1}{xy} + \frac{1}{y^2} = 91$ ① ÷ ②

④ $\frac{1}{x^2} - \frac{2}{xy} + \frac{1}{y^2} = 1$ ② squared

⑤ $\frac{3}{xy} = 90$, or $\frac{1}{xy} = 30$ ③ − ④

Complete by the method of Example 2.

***8.** Solve: ① $x + y + xy = 35$ ② $x^2 + y^2 - x - y = 62.$

Plan of solution: ③ $2x + 2y + 2xy = 70$ ① × 2

④ $x^2 + 2xy + y^2 + x + y - 132 = 0$ ② + ③

or $(x + y)^2 + (x + y) - 132 = 0$

⑤ $[(x + y) + 12][(x + y) - 11] = 0$. Factoring

⑥ $x = -y - 12$

⑦ $x = 11 - y$

Complete by substituting ⑥ and ⑦ in ①, group answers, and check:

***9.** $x^2 + y^2 + x + y = 18 \qquad xy = 6$

10. $x^2 + y^2 = 25 \qquad xy = 12$

11. $2x^2 - 3xy - 2y^2 = 6 \qquad x^2 - xy + 2y^2 = 4$

12. $x^2 + xy = 10 \qquad y^2 + xy = 6$

13. $3x^2 + y^2 = 7 \qquad x^2 + 4xy = 9$

14. $x^2 - 3xy = -8 \qquad 3y^2 + xy = 20$

***15.** $x^2 + y^2 = m \qquad xy = n$

***16.** $x^2 + y^2 - x - y = 62 \qquad x + y + xy = 47$

***17.** The sum of the squares of two numbers is 44 greater than the sum of the numbers, and the product of the numbers plus their sum is 23. Find the numbers.

PART II

EXERCISES

PART II

EXERCISES

As a student of algebra or of any other science you need both intelligence and skill. Part I is largely devoted to the development of understanding and Part II to the development of skill.

For a list of the exercises and a statement of aims see the Table of Contents. See also the Preface, page vi.

EXERCISES FOR CHAPTER I

THE NATURE AND PURPOSE OF ALGEBRA

The tests of Chapter I will indicate which of the following Chapter I exercises you need. Remember that many failures in algebra are due to lack of mastery of the simple operations.

Exercise 1. Addition of Monomials

(See page 15.)

1, A. Uniting Similar Terms

1. The "short and over" account of a cash register showed, day by day, the following items. Was the register "short" or "over" for the week? How much? Short, \$1.15; over, \$0.83; over, \$1.07; short, \$0.03; short, \$1.00; over, \$2.18.

Unite similar terms.

2. $4 - 5 + 8 - 6 + 0 - 2$ **3.** $x - 3 + 2x - 5 - x + 4$

4. $a + b + c - a + b - c$ **5.** $a - x - a + x$

6. $x_1 + 2x_2 + 3x_3 - x_1 + 2x_2 - 3x_3$

7. $x - 3x^2 - 4x^3 + 3x^2 - x - 5x^3$

8. $2a - a' + 3a - 2a' - 5a + a'$

9. $a + 3a + a^3 + a_3 - a_3 - 2a - 3a^3$

10. $2ab^2 - 2a^2b - 2a^2b - 2a^2b$

11. $x^2 + 5x + 2 - x^2 - 4x - 8 - 10x$

12. $2(a + b) + 3(a + b) - (a + b)$

13. $5(x - y) - 7(x - y) - (x - y)$

1, B

Unite similar terms and inclose each polynomial coefficient in a parenthesis.

1. $ax + ab \equiv (\ ?\ )a$
2. $ax + 2by + y$
3. $ay - by$
4. $ax - ay - az$
5. $\frac{1}{2}ah + \frac{1}{2}bh$
6. $a^3 - a^2 - a$
7. $3a^2x - 2ax - a$
8. $6x^4y^2 - 6x^3y^3 + 9x^2y^5$
9. $25a^2b^2 - 30ab^2 - ab$
10. $abc + ac + wac - wa^2$
11. $60ab^2m + 84bc^3m + 36ac^4m$
12. $a^2b^2c^3d^4 - ab^3c^3d^4 - ab^2c^4d^4$
13. $c^2 - 2c$
14. $ac - 2c$
15. $abc - 2efc$
16. $abc - 2bc$
17. $abc - 2abc$
18. $abx - 2aby$

1, C

Errors are very common in the kind of addition called for below. Repeat this exercise whenever you make such errors. Unite:

1. $3(a + b) + 4(a + b)$
2. $5(a - b) - 6(a - b)$
3. $3(a + c) + (a + c)$
4. $5(a - c) - (a - c)$
5. $a(x - y) + b(x - y)$
6. $a(x - y) + (x - y)$
7. $r(s - t) - (s - t)$
8. $2a(m + n) - 3b(m + n)$
9. $3t(r + s) + 2x(r + s)$
10. $5a(c - d) - (c - d)$

11. $a(b-1)+(b-1)$

12. $a(b-1)+(1-b)$

13. $2\,m(x+y)-n(-x-y)$

14. $x(b-c)-y(b-c)$

15. $s(a-c)-2\,t(c-a)$

16. $2\,a^2(2\,a-1)-3(1-2\,a)$

17. $4\,m(2\,a-b)-(b-2\,a)$

18. $5\,x^2(2\,a+c)-2\,ax^2-cx^2$

19. $(a-b)x+(a-b)$

20. $(a-b)x+x$

21. $(x-x_1)a-a$

22. $(x-x_1)bc-3\,bc$

1, D

Practice the following additions until you can perform them rapidly and without errors:

1.	$+8$	$+3$	$+4$	5	7	0	9		
	$+5$	$+6$	$+4$	6	0	8	1		
2.	$4\,a$	$3\,a$	$5\,a$	$+7\,a$	$+8\,a$	0	$8\,a$		
	$2\,a$	$5\,a$	$3\,a$	$+6\,a$	$+9\,a$	$5\,a$	0		
3.	$5\,b$	b	$+b$	1	0	b	$+a$		
	b	$7\,b$	$+b$	b	b	0	$+b$		
4.	a	$+b$	$2\,b$	a	c	$+c$	$+3\,c$		
	n	$+a$	a	$2\,b$	c	$+1$	$+2\,c$		
5.	-7	-3	-4	-5	-7	0	-5		
	$+5$	$+6$	$+4$	6	0	0	-5		
6.	$+8$	4	3	4	5	0	$+7$	$\frac{1}{2}$	$\frac{3}{4}$
	-7	-9	-3	-5	-4	-3	-7	$.5$	$.75$
7.	-1.5	-1.0	-1.4	$+7.8$	-1	$+2$	$+\frac{1}{2}$		
	$-\ .6$	$-\ .5$	-2.4	3.5	$-\frac{1}{2}$	$-\frac{1}{3}$	$+\frac{1}{3}$		

8.
$$\begin{array}{rrrrrrr} x & a & 3\,a & .4\,a & .5\,a & .07\,b & b \\ .6\,x & .25\,a & -\,.3\,a & a & -\,a & b & -\,.02\,b \\ \hline \end{array}$$

9.
$$\begin{array}{rrrrrrr} -\,2\,a & -\,3\,a & -\,7\,a & -\,3\,x & -\,8\,x & -\,x & x \\ 5\,a & -\,4\,a & -\,7\,a & -\quad x & +\quad x & 0 & -\,2\,x \\ \hline \end{array}$$

10.
$$\begin{array}{rrrrrrrr} ax & bx & ax & x & x & -\,ax & 0 & 4\,x^2y \\ ax & -\,2\,bx & bx & ax & x & -\quad 1 & -\,bx & -\,5\,x^2y \\ \hline \end{array}$$

11.
$$\begin{array}{rrrrr} (a+b)x & (a-b)x & (c+d)x & (c+d)x & (a-b)x \\ cx & (a+b)x & x & (c+d)x & a-b \\ \hline \end{array}$$

***12.†**
$$\begin{array}{rrrr} rs(a_1-a_2) & ax^2(r-s) & 2\,ab^2(x-y) & (a_2-a_3)a^2 \\ -\,2(a_1-a_2) & ax^2(r+s) & -\,c(y-x) & (a_4-a_3)a^2 \\ \hline \end{array}$$

Exercise 2. Subtraction of Monomials

(See page 16.)

1. How many years was it from 10 B.C. to 15 A.D.? From 44 B.C. to 1492 A.D.? Write each of these questions as an example in the subtraction of signed numbers.

2. State and explain the law of signs for subtraction.

3. In Exercise 1, D, subtract each lower term from the term above it.

4. In Exercise 1, D, subtract each upper term from the term below it.

Exercise 3. Multiplication of Monomials

(See page 17.)

1. If distance east is called positive and distance west negative, and if time to come is called positive and time past nega-

† Starred examples may be omitted in a minimum course or they may be used as honor work for more able pupils. (See the time schedule in the preface, p. viii.)

tive, how can we indicate the position of a train traveling eastward 30 miles an hour 3 hours after it passes the observer? Two hours before it reaches him? Answer the same questions for a westbound train traveling 25 miles an hour. Use a drawing if necessary. Represent each of these four questions as an example in the multiplication of signed numbers.

2. State and explain the law of signs for multiplication.

3. In Exercise 1, D, multiply each upper term by the term below it.

Simplify each fraction below by multiplying numerator and denominator by a suitable multiplier:

4. $\dfrac{1\frac{1}{2}}{4\frac{2}{5}}$ $\qquad \dfrac{a+\frac{1}{2}}{b+\frac{2}{3}}$ $\qquad \dfrac{4\frac{1}{2}\,a}{7\frac{1}{3}\,b}$ $\qquad \dfrac{\dfrac{a}{x}}{\dfrac{b}{x^2}}$ $\qquad \dfrac{\dfrac{a+b}{x}}{\dfrac{c+d}{y}}$

5. $\dfrac{a+b+\frac{1}{2}}{a+2\,b+\frac{1}{4}}$ $\qquad \dfrac{x+\dfrac{y}{2}}{2\,y-\frac{2}{3}}$ $\qquad \dfrac{\dfrac{a}{x}+\dfrac{b}{y}+c}{\dfrac{a}{y}-\dfrac{b}{2\,x}}$ $\qquad \dfrac{\dfrac{1}{5}+\dfrac{2}{a}+\dfrac{3}{b}}{2\frac{2}{3}-\dfrac{1}{2\,b}}$

Exercise 4. Division of Monomials

(See page 18.)

1. State the law of signs for division. Show that it follows from the law of signs for multiplication.

2. In Exercise 1, D, divide each upper term by the term below it. Omit the examples in which the divisor is zero. When the quotient is not an integer, express it as a fraction in lowest terms.

Exercise 5. The Order of Operations

(See page 18.)

Find the value of each of the following expressions:

1. $8 \div 2 + 1 - 3$ **2.** $12 \div 2 \cdot 4 - 3 \cdot 2$ **3.** $4(-2) - 2 \div 2$

4. $(-6)(-2) + (-4) + (-2)$ **5.** $\dfrac{6}{-2} + \dfrac{-8}{4} + (-3)$

6. $3(-2) + \dfrac{-6}{-3} - (-3) + 6.0$ **7.** $15 - (-4.4) + (-3.2)$

8. $18 \div 12 + (4 \cdot 1.5) - (-.5)$

9. $\dfrac{6+9}{3}$ **10.** $\dfrac{6 \times 9}{3}$ **11.** $\dfrac{8+4}{3}$

12. $5(7-3) - 2(3-7)$ **13.** $(15 - 2 \cdot 3)(5-7) \div \left(\dfrac{-14}{-2}\right)$

14. $(8 \div 4 \times 3 - 4 + 5) - (21 \div 7 \cdot 2 - 10) \cdot 2$

Perform the operations indicated:

15. $5a + 10ab \div 2b$ **16.** $2ab \cdot a + 3a^2b$

17. $24x^3y^2 \div 6xy^2 + 5x^2y$ **18.** $3a^2 + 4a(a-1) - 7a^2$

19. $x^2 - 2x^2b \div (-4x^2)$ ***20.** $x^2 - 2x^2y \div y - 4x^2$

***21.** $50a^3b^3 - 32a^5b^2 \div 4a^3b \times 5ab^2$

***22.** $5a^2b + 2a^2b^2 \div ab \cdot a - 4a \cdot ab$

Exercise 6. Substitution and Evaluation

(See page 19.)

Object: To test your understanding of algebraic symbols and your ability to use signed numbers.

When $a = 4$, $b = 3$, $c = 1$, and $d = 0$, find the value of:

1. $2a + 3(b - c)$ **2.** $(2a + 3)(b - c)$

3. $\dfrac{2a^2+2c}{2b^2-c}$

4. $\dfrac{(a+b)(a-b)+1}{2ac^2-3d}$

5. $\sqrt{3a^2+5b+1}$

6. $3c\sqrt{ab^2}-(2a-3b)^2$

7. $(2a+c)\sqrt{3b^3c^4}$

8. $\sqrt[3]{(a+b+c)^2}$

9. $\sqrt{a^2+b^2}$

10. $\sqrt{a^2+2ab+b^2}$

In examples 1–5, substitute 5 for a, -1 for b, $\frac{1}{2}$ for c, and -2 for d, and find the resulting values.

11. If $a=2$, $b=-13$, and $c=20$, find the value of

$$\frac{-b-\sqrt{b^2-4ac}}{2a}$$

12. Show by numerical substitution that $-(b-a)\equiv a-b$

$$\frac{-a}{-b}\equiv\frac{a}{b}\equiv-\frac{-a}{b} \qquad \frac{a^3+b^3}{a+b}\equiv a^2-ab+b^2$$

13. If $x=2$, prove that $5(3x-4)-2(1-x)=12$.

14. If $x=-4$, prove that $3x^2-(x-1)^2=23$.

15. Does $3(5a-2)+14=4a^2+2a+11$, if $a=3$? if $a=0$? if $a=2$?

16. If $x=\dfrac{-2}{5}$, does $\dfrac{20x^2}{7}-x=\dfrac{6}{7}$?

17. For what integral value of x, between 5 and 8, does $\dfrac{2x-1}{2}-\dfrac{1}{3}=5+\dfrac{x}{6}$?

18. If $x=4\frac{12}{25}$, does $\dfrac{5}{14}=\dfrac{8}{5x}$?

19. Does $\dfrac{x-2}{x+4}-\dfrac{x^2+8}{x^2+2x-8}$ equal zero if $x=2$? if $x=-1$?

*20. If $x=-3\frac{1}{2}$, does $\dfrac{x-1}{x^2+2}=\dfrac{2x+4}{x+13}$?

*21. When $x = 2\frac{2}{3}$ and $y = -4\frac{1}{5}$, find the value of $\frac{2x - 3y^2}{x - y}$. (Simplify the complex fraction as suggested for examples 4 and 5 of Exercise 3 on page 351.)

*22. When $x = 1\frac{1}{3}$ and $y = -2\frac{1}{2}$, find the value of $\frac{3xy - y^2}{x - 2y}$.

Exercise 7. Linear Equations in One Unknown

7, A

(See page 27.)

1. State the fundamental law for the transformation of equations. Solve and check each equation below. In one or more of the solutions explain each transformation without using the words "transpose" or "cancel."

2. $5x - 7 - (x + 5) = 3x + 1$

3. $18 = 4a - (a - 9)$

4. $3x - (x - 5) = 16 - (3 + 2x)$

5. $23 = 25 - 2(a - 4)$

6. $4(2x - 5) = 5(4x - 7) + 9$

7. $2(x - 2) = 3(x - 5) + 11$

8. $9 = 7(x - 3) - 2(4 + x)$

9. $2(4 - x) - 3(x - 6) = 0$

10. $24 - 3(8 - x) = 5(x - 3) - 7(6 - x) + 3$

11. $3(x - 3) = x + 2(x - 2) - 5$

7, B. Fractional Equations with Monomial Denominators

(See page 28.)

1. $\frac{2x}{9} + \frac{x}{6} = \frac{x}{18} + \frac{1}{3}$

2. $\frac{4}{x} = \frac{8}{5}$

3. $\frac{5}{7} = \frac{21}{x}$

4. $\frac{x + 3}{2} + 2x = \frac{x + 5}{3} + 15$

5. $\frac{x + 1}{2} + \frac{x + 3}{4} = x - 2$

6. $\frac{x + 5}{3} + \frac{x - 5}{2} = \frac{3x}{4}$

7. $\frac{a + 18}{4} - \frac{3}{7}(a - 3) = 4$

8. $\frac{11x + 19}{6} - \frac{6x - 5}{3} = \frac{6x + 1}{6}$

9. $\dfrac{8x-3}{7} - \dfrac{4x-7}{5} = 5x - 13$

10. $\dfrac{x+3}{5} - \dfrac{2x-13}{11} = \dfrac{x}{3} - 2$

11. $\dfrac{4y+3}{5} - 4 - \left(y - \dfrac{2y-2}{6}\right) = 0$

12. $\dfrac{2}{3} - \dfrac{1}{3}\left(6 - \dfrac{3y}{4}\right) + \dfrac{y}{6} = \dfrac{1}{2}\left(4 - \dfrac{2y}{3}\right) - \dfrac{1}{3}$

13. $\dfrac{2a+0.27}{5} = \dfrac{a-0.16}{3}$ *14. $\dfrac{4.6y-3.6}{7} = \dfrac{2.3y-2.7}{9}$

Exercise 8. Literal Equations in One Unknown

(See page 28.)

Solve for x. If necessary use parentheses in expressing the coefficients of x. Check by substituting each root in the original equation, or by numerical substitution.

1. $ax + b = c$ 2. $(a+b)x = c - d$ 3. $ax + bx = c$

4. $cx - x = 7$ 5. $a^2x = 3 + x$ 6. $ax - a = b - bx$

7. $\dfrac{x}{a} = b - 5$ 8. $\dfrac{a}{x} = \dfrac{c}{d}$ 9. $\dfrac{a}{x} - b = \dfrac{c}{3x}$

10. $\dfrac{b}{x} = \dfrac{c+d}{a}$ *11. $\dfrac{x}{a} + \dfrac{x}{b} - \dfrac{x}{c} = 1$

Exercise 9. Linear Equations in Two Unknowns

9, A

(See page 30.)

1. What does it mean to *solve* a pair of linear equations in two unknowns? Solve and check:

2. $\begin{aligned} x + y &= -3 \\ x - 3y &= 5 \end{aligned}$ 3. $\begin{aligned} x + 3y &= 7 \\ 2x + 5y &= 12 \end{aligned}$

4. $2x + 4y = 16$
$3x + 5y = 21$

5. $x + 2y = 12$
$x - 2y = -8$

6. $3x + 4y = 13$
$-4x - y = -13$

7. $2x + 5x_1 = 29$
$3x - 2x_1 = -4$

8. $3x + y = 11$
$10x - 3y = 5$

9. $3x - y = 11$
$8x - 7y = 51$

10. $.5x + .4y = .13$
$.7x + .3y = .13$

11. $.3x + .2y = 9.5$
$.2x + .3y = 10.5$

9, B

1. $\dfrac{x}{y-3} = 4$
$\dfrac{x+2y}{9} = y - 3$

2. $\dfrac{x + 2x_1}{2x} = 1 + \dfrac{1}{x}$
$3x_1 - \dfrac{5x}{6} = \dfrac{4x+1}{3}$

3. $\dfrac{a}{2b} = \dfrac{1}{8b} + \dfrac{2}{5}$
$3a - 2b = a - \frac{3}{4}$

4. $\dfrac{.2x_1 + .5}{1.5} = \dfrac{.49x - .7}{4.2}$
$\dfrac{.5x - .2}{1.6} = \dfrac{41}{16} - \dfrac{1.5x_1 - 11}{8}$

5. $\dfrac{x - x_1}{2} - \dfrac{x - 3x_1}{5} = x_1 - 3$
$\frac{3}{4}(x - x_1) + \frac{5}{6}(x + x_1) - 18 = 0$

6. $\dfrac{x - x_1}{2} - \dfrac{x - x_1}{3} = 8$
$\dfrac{x + x_1}{3} + \dfrac{x - x_1}{4} = 11$

7. $\dfrac{x}{4} - \dfrac{y}{3} - \dfrac{x+y}{12} = 3$
$\dfrac{x}{3} + \dfrac{3y}{2} - \dfrac{y - 2x}{6} = -6$

8. $\dfrac{2}{a} + \dfrac{4}{b} = 10$
$\dfrac{6}{a} - \dfrac{2}{b} = 10$

9. $\dfrac{7}{x} + \dfrac{8}{y} - 30 = 0$
$\dfrac{8}{x} + \dfrac{7}{y} - 30 = 0$

10. $\frac{9}{x} - \frac{10}{y} = 1$

$\frac{12}{x} + \frac{15}{y} - 7 = 0$

11. $\frac{1}{x} - \frac{2}{y} = -\frac{3}{4}$

$\frac{2}{x} + \frac{3}{y} = \frac{4}{5}$

12. $\frac{10}{x} + \frac{9}{x_1} = 8$

$\frac{8}{x} + \frac{3}{x_1} = 5$

13. $\frac{1}{2w} + \frac{1}{3v} = 8$

$\frac{1}{4w} + \frac{1}{9v} = \frac{10}{3}$

9, C.

Solve for x and y

1. $x + y = r$
   $x - y = t$

2. $ax + by = 1$
   $x + y = 2$

3. $ax + by = c$
   $ax - by = n$

4. $ax + 2y = c$
   $2x + ay = b$

5. $rx + sy = t$
   $sx - ry = v$

6. $\frac{x}{a} + \frac{y}{b} = 1$
   $\frac{x}{b} - \frac{y}{a} = \frac{1}{2}$

*7. $a_1x + b_1y = k_1$
   $a_2x + b_2y = k_2$

9, D

Special exercise for pupils who are weak in addition and subtraction.

In Exercise 9, A, add each upper equation to the equation below it.

In Exercise 9, A, subtract each lower equation from the equation above it; also subtract each upper equation from the equation below it.

Exercise 10. Multiplication of Binomials

(See page 32.)

Multiply with or without pencil:

1. $(x+3)(x+8)$
2. $(x+4)(x-3)$
3. $(a-9)(a+4)$

4. $(b+7)(b-8)$ 5. $(x+a)(x+b)$ 6. $(x+y)(x+y)$
7. $(x+3)(x+3)$ 8. $(x+4)^2$ 9. $(x+12)^2$
10. $(x+5)^2$ 11. $(x-a)(x-a)$ 12. $(x+a)(x-a)$
13. $(x-b)(x+b)$ 14. $(x-3)(x+3)$ 15. $(x+5)(x-5)$
16. $(9+x)(9+x)$ 17. $(9-x)(9-x)$ 18. $(9-x)(9+x)$
19. $(3+x)(x+4)$ 20. $(5+x)(x-5)$ 21. $(x-2)(8+x)$
22. $(x-7)(9+x)$ 23. $(7-x)(x-8)$ 24. $(8-x)(x+6)$
25. $(7x-8)(x+2)$ *26. $(30+2)^2$ *27. $(32)^2$
*28. $(70-1)^2$ *29. $(69)^2$ *30. $(29)^2$
*31. $(40-2)(40+2)$ *32. $(38)(42)$ 33. $(59)(61)$

Exercise 11. Factoring Quadratic Expressions of the Types $x^2 \pm ax \pm b$ $a^2 \pm 2ab + b^2$ and $a^2 - b^2$

(See page 33.)

11, A

Factor the following expressions. If a monomial factor is common to all the terms of any expression, remove that factor first.

1. x^2+8x+7 2. $x^2+7x+12$ 3. x^2+4x+4
4. $x^2+10x+16$ 5. x^2+7x+6 6. $10+7x+x^2$
7. $ax^2+17ax+60a$ 8. $60b+23bx+bx^2$
9. $cx^2+19cx+60c$ 10. $x^2+61x+60$
11. $3x^2-12x+9$ 12. $12-14x+2x^2$
13. $2x^2-4x+2$ 14. $x^2-8x+15$
15. $3x^2-24x+21$ 16. $16-8x+x^2$
17. $100-20x+x^2$ 18. $x^2-25x+100$
19. $x^2-52x+100$ 20. $x^2-(a+b)x+ab$

21. $x^2 - (r + s)x + rs$ 22. $ax^2 + 7\,ax - 18\,a$

23. $abx^2 - 7\,abx - 18\,ab$ 24. $x^2 - 8\,x - 20$

25. $x^2 + 8\,x - 20$ 26. $4\,x^2 + 4\,x - 80$ 27. $-20 - x + x^2$

28. $x^2 - 7\,x - 30$ 29. $x^2 + 7\,x - 30$ 30. $2\,x^2 - 2\,x - 60$

31. $x^2 - (a - b)x - ab$ 32. $x^2 - (r - s)\,x - rs$

33. $a^2 - 25$ 34. $36 - x^2$ 35. $2\,a^2 - 2\,b^2$

36. $(x + 3)^2 - 4$ 37. $(x + 5)^2 - 64$ 38. $(x - 1)^2 - 25$

39. $(x - 7)^2 - 36$ 40. $(a - 5)^2 - 1$ 41. $(s - t)^2 - 9$

42. $16 - (x - 5)^2$ 43. $x^2 - 6\,x + 9$ 44. $x^2 + 6\,x + 9$

45. $x^2 - 9$ 46. $ax^2 + 10\,ax + 25\,a$ 47. $4\,x - 4\,x^2 + x^3$

48. $x^3 - 2\,x^2 + x$ 49. $60 - 30\,x + 3\,x^2$ 50. $x^2 - 25$

51. $2\,x^2 - 50$ 52. $2\,x^2 - 10\,x - 50$ 53. $50 - 24\,x + 2\,x^2$

54. $(a - b)^2 - 2(a - b) + 1$ 55. $(r - s)^2 - 6(r - s) - 16$

11, B

Which expressions in Exercise A are perfect trinomial squares?

Exercise 12. Multiplication of Polynomials

(See page 33.)

Multiply; use pencil only when necessary; check some of the results by numerical substitution. If you know how, check the answers to 20–22 by long division.

1. $(2\,x + 5)(4\,x + 7)$ 2. $(3\,x + 2)(5\,x + 6)$

3. $(4\,x + 3)(5\,x + 2)$ 4. $(8\,x - 7)(6\,x - 1)$

5. $(3\,x + 5)(3\,x + 5)$ 6. $(3\,x - 5)(3\,x - 5)$

7. $(3\,x + 5)(3\,x - 5)$ 8. $(4\,x + 7)^2$

9. $(4\,x - 7)(4\,x + 7)$ 10. $(4\,x - 7)^2$

11. $(6 - 5\,x)^2$ 12. $(6 + 5\,x)^2$

13. $(6-5x)(6+5x)$

14. $(3x+5)(7-4x)$

15. $(8x-7)(5-6x)$

16. $(a+b+c)^2$

17. $(a-b+c)^2$

18. $(\frac{1}{2}a+x)(2a-x)$

19. $(\frac{1}{2}x+\frac{1}{3}y)(\frac{1}{2}x-\frac{2}{3}y)$

20. $(x^2-5x+7)(3x-2)$

21. $(a^2-ab+b^2)(a+b)$

22. $(a^2+ab+b^2)(a+b)$

Exercise 13. Factoring Quadratic Expressions of the Type $ax^2 \pm bx \pm c$

(See page 33.)

Factor if possible; remove monomial factors first. Check some of the results by numerical substitution.

1. $2a^2+5a+2$

2. $6x^2+13x+6$

3. $3x^2+17x-6$

4. $6x^3+37x^2+6x$

5. $-12x+13x^2+4x^3$

6. $20x^2+37xy+15y^2$

7. $6a^4-11a^3b-7a^2b^2$

8. $-6y^2-10xy+4x^2$

9. $4x^2+12xy+9y^2$

10. $4ax^2-12axy+9ay^2$

11. $4x^2-21xy-18y^2$

12. $16ax^2+40axy+25ay^2$

13. $16bx^2-40bxy+25by^2$

14. $16ax^2-25ay^2$

15. $64a^2+112ab+49b^2$

16. $64a^3-112a^2b+49ab^2$

17. $-49ab^2+64a^3$

***18.** $\frac{1}{9}x^2+\frac{1}{3}xy+\frac{1}{4}y^2$

***19.** $\frac{1}{9}x^2-\frac{1}{3}xy+\frac{1}{4}y^2$

***20.** $\frac{1}{9}x^2-\frac{1}{4}y^2$.

21. Which of the preceding expressions are perfect trinomial squares?

Exercise 14. Solving Quadratic Equations by Factoring

(See page 34.)

Solve and check:

1. $x^2-9x+20=0$

2. $x^2-15x+56=0$

3. $40 - 13x + x^2 = 0$

4. $x^2 = 13x - 42$

5. $x^2 = 23x - 126$

6. $40 = 13x - x^2$

7. $36 = 15x - x^2$

8. $8x^2 - 22x = -15$

9. $3x^2 + 13x - 56 = 0$

10. $56 - 10x = x^2$

11. $0 = c - c^2 + 56$

12. $3x^2 - 11x - 20 = 0$

13. $2x^2 - 9x - 35 = 0$

14. $x^2 - 6x = 0$

15. $2x^2 - 5x = 0$

16. $x(5x - 23) = 42$

17. $\frac{1}{2} - \frac{5}{6x^2} = -\frac{7}{12x}$

18. $\frac{6}{x-2} = x - 7$

19. $\frac{6}{5+s} = 6 + s$

20. $x - \frac{6}{x-5} = 10$

21. $\frac{4}{5x - 0.5} = 10x + 1$

22. $\frac{x}{x-2} - 5 = 4$

23. $\frac{8}{x+1} + 5 = \frac{12}{x-2}$

24. $3(x+1)(x-1) = 2(27 - 5x)$

25. $\frac{1}{1-2x} + \frac{1}{1-4x} = \frac{5}{6}$

26. $\frac{3}{y+4} - \frac{4}{y-1} = 4$

27. $\frac{x}{x-2} + \frac{x-2}{x} = \frac{5}{2}$

28. $\frac{x}{x+2} = \frac{x+2}{2x} - \frac{1}{2}$

29. $\frac{1}{x+2} = \frac{x}{x-2} + \frac{4}{3}$

*30. $\frac{x+3}{4x+5} = \frac{3x+1}{5x+4}$

*31. $\frac{x+4}{x-4} + \frac{x-4}{x+4} = \frac{82}{9}$

*32. $\frac{1}{6} = \frac{1+x}{2+x} - \frac{3-x}{5-x}$

33. $\frac{y-1}{y+3} + \frac{y-5}{7} = \frac{2y-3}{14}$

*34. $\frac{a-3}{3} - \frac{3a+2}{2a+3} = \frac{2a}{6}$

Exercise 15. Solving Quadratic Equations by Formula. Exact Answers

(See page 35.)

Arrange in suitable form and then solve by formula at least ten of the equations of Exercise 14.

Exercise 16. Extracting Square Roots of Numbers

(See pages 36, 37, 39.)

16, A. Exact Square Roots

Find the square roots of the numbers below. Check by multiplication.

1. 1156 **2.** 8464 **3.** 12.25 **4.** 1225

5. 146.41 **6.** 3364 **7.** 33.64 **8.** 84.64

9. 6889 **10.** 94.09 **11.** .7569 **12.** 4.2025

13. 402.8049 **14.** .283024 **15.** .00120409

16. Define square root, and show that it follows from the definition that $(\sqrt{5})^2 \equiv 5$.

16, B. Approximate Square Roots

Extract the square root of each number below to the nearest third figure or to the nearest fourth figure as required.

1. 7.45 **2.** 88.6 **3.** 876 **4.** 5580

5. 5.58 **6.** 55.8 **7.** 433 **8.** 4.33

9. 43.3 **10.** 2 **11.** 3. **12.** 5

13. 5000 **14.** 5976 **15.** 8312 **16.** .2469

17. .0365 **18.** .4231 **19.** 17 **20.** .41444

Exercise 17. Solving Quadratic Equations with Irrational or Imaginary Roots

(See page 41.)

Solve by formula, and check. Extract roots to three-figure accuracy or to four-figure accuracy as directed. If you think that the roots of any equations are imaginary numbers, indicate that fact and tell why you think so. You need not check imaginary roots.

1. $x^2 + 6x - 3 = 0$

2. $x^2 + 20x - 98 = 0$

3. $x^2 + 20x + 98 = 0$

4. $2x^2 + 8x = 9$

5. $5x^2 + 8x = 1$

6. $3x^2 + 9 = 5x$

7. $x - \frac{7}{3} = \frac{1}{x+2}$

8. $1 + \frac{4x}{(x+1)^2} = \frac{2}{x+1}$

9. $x^2 - 6 = 0$

10. $3x^2 + 10 = 0$

11. $10x^2 = 15x$

Exercise 18. The Solution of Equations

(See page 41.)

Solve the following equations and verify all roots except those which are imaginary. Extract approximate square roots to the nearest fourth figure. Discard any root which does not check.

1. Name the three kinds of equations which are discussed in Chapter I.

2. State the fundamental law for the transformation of equations.

3. Mention at least one of the purposes which equations serve.

4. $2x^2 - x - 15 = 0$

5. $2x^2 - x - 5 = 0$

6. $2x^2 - x + 15 = 0$

7. $3x - y = 6$
$x + 9y = 86$

8. $0.8a + 0.1b = 0.19$
$0.6a + 0.9b = 0.39$

9. $\frac{a}{bx + c} = g$

10. $\frac{3}{x}+\frac{4}{y}=\frac{7}{24}$

$\frac{8}{x}-\frac{6}{y}=9$

11. $\frac{4}{a+1}+\frac{5}{b-2}=9$

$\frac{5}{a+1}-\frac{3}{b-2}=2$

***12.** $\frac{a}{bx}+\frac{b}{ay}=a+b$

$\frac{b}{x}+\frac{a}{y}=a^2+b^2$

***13.** $\frac{a+x}{a+b}+\frac{ax}{b}=\frac{b}{a}$

14. $\frac{3}{5x-1}=\frac{1}{2x-1}$

***15.** $\frac{1}{x+4}-\frac{7}{(x-3)(x+4)}=\frac{2}{x+2}$

16. $\frac{y-1}{y-\frac{4}{3}}=\frac{y+\frac{1}{3}}{y-\frac{2}{3}}$ First multiply numerator and denominator by 3.

17. $ax+y=r$

$x+cy=s$

18. $(x-5)(x-10)=4(x-10)$

19. $\frac{x(2x-5)}{30}=\frac{x}{5}-\frac{1}{6}$

20. $(x+1)(2x+3)=x^2-11$

***21.** $\frac{x-1}{2}-\frac{x-1}{2x+1}=\frac{x-1}{6}+\frac{x-3}{3}$

22. $\frac{x+2}{4}+\frac{x}{2}=\frac{3x}{4}-\frac{2}{3x+5}$

23. $2x[3-(5x+10-6x)]=4x-(7-2x)+3$

***24.** $\frac{w-2}{3}-\frac{w_1+1}{5}=3$ $\frac{2w-11}{5}+\frac{9+2w_1}{3}=0$

Exercise 19. Formulas

(See page 43.)

19, A. Evaluation

1. What purposes do formulas serve? Why is it important to know how to evaluate in a formula?

Find the missing values:

2. Horse power of a gas engine: $H = \dfrac{nd^2s}{10}$.

H = rated horse power; n = number of cylinders; d = internal diameter (bore) in inches; s = length of stroke in inches.

	n	d	s
Ford	4	$3\frac{3}{4}$	5
Buick	6	$3\frac{3}{4}$	5
Packard	8	$3\frac{1}{2}$	5
Cadillac	16	3	4

3. A coefficient in the binomial formula:

$$c = \frac{n(n-1)(n-2)(n-3)}{1 \times 2 \times 3 \times 4}$$

I. $n = 4$ II. $n = 8$ III. $n = 12$ IV. $n = 20$

4. The sum of an arithmetic series: $S = \dfrac{n}{2}(a + l)$

	S	n	a	l
I		8	0	10
II		12	1	−12
III	30		2	8
IV	45	5	3	
V	350	20		30

5. The last term of an arithmetic series: $l = a + (n - 1)d$

	l	a	n	d
I		7	20	4
II		2	16	− 5
III	346	1	24	
IV	120		18	7
V	45	− 3		4

6. The sum of a geometric series: $S = \dfrac{rl - a}{r - 1}$

	S	a	r	l
I.		1	2	64
II.		1	$\frac{1}{4}$	$\frac{1}{64}$
III.		2	$1\frac{1}{2}$	$\frac{81}{8}$
IV.		.08	1.3	.1352
V.	468	3	5	

7. Interest: $i = prt$.

I. What is the interest on \$500 at 4% for $\frac{1}{3}$ year?

II. How many months' interest on \$600 at 3% will amount to \$6?

III. The interest on \$400 for 6 months was \$14; what was the rate?

8. Area of a ring: $A = \pi r^2 - \pi r_1{}^2$. $r = 14.34$, $r_1 = 11.34$. Give answer to four-figure accuracy. Compare direct substitution in the formula with substitution after making the transformation $A = \pi(r - r_1)(r + r_1)$.

9. The area of a trapezoid: $A = \frac{1}{2}\,h(a + b)$.

	A	h	a	b
1.		10	13	11
2.	150	12	10	
***3**.		2.13	3.55	4.57
***4**.	70.8	10.6		7.98

(In 3 and 4 work to three-figure accuracy.)

***10.** A roof which is a trapezoid in shape measures 42′ at the eaves, 27′ at the ridge, and 20′ from eaves to ridge. How many squares does it contain? (By a "square" a roofer means a square 10′ on a side.)

***11.** A tomato can holds 1 lb. 3 oz., and a second can of the same height is to be made large enough to hold 2 lb. 1 oz. The diameter of the small can is 3.3″; find the diameter of the large

can to the nearest tenth of an inch. The formula for the volume of a cylinder is $V = \pi r^2 h$.

19, B. Transformations of Formulas

1. Under what conditions is it helpful to transform a formula before evaluating?

2. Solve each formula of Exercise A for each letter in turn if you can. If necessary, repeat exercise 8, page 355.

Solve for each letter in turn:

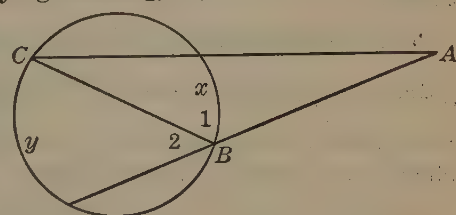
3. $c = 2\pi r$ **4.** $f = \frac{9}{5}c + 32$ **5.** $a = p(1 + rt)$

6. $c^2 = a^2 + b^2$ **7.** $s = \frac{1}{2}gt^2$

8. $\dfrac{D}{d} = q + \dfrac{r}{d}$ (This formula shows the relation between the dividend, divisor, quotient, and remainder in division. State in words the meaning of the formula and of each transformed formula.)

9. $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$ **10.** $i = \dfrac{e}{R + nr}$ **11.** $\dfrac{a}{b + c} = \dfrac{d}{e}$

12. In the accompanying drawing, A, B_1, B_2, C, x, and y represent the number of degrees in the angles and arcs designated; and it is known that $B_2 = C + A$, and that $C = \frac{1}{2}x$, and $B_2 = \frac{1}{2}y$. Find A in terms of x and y.



***13.** Transform the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ into a formula for the solution of a quadratic equation in which the coefficient of x^2 is 1.

19, C. Formulas, Miscellaneous

1. What is the cost in dollars of x articles at y cents each?

2. How long will it take a man to walk m miles if he walks at the rate of b miles in c hours? (*Either plan your work or check it by means of numerical illustration.*)

3. Using the information below, fill the blanks in the table:

	Principal	*Rate*	*Annual Income*
First investment.......			
Second investment.....			
Total...............		omit	

A total of t dollars is invested, d dollars at p per cent and the rest at p_1 per cent.

***4.** A man's total income from two investments was t dollars; on the first investment of a dollars, he received p %. What was the income from the second investment? If the rate on the second investment was p_1%, what was the amount of this investment?

***5.** Each side of a triangle is s; express the altitude in terms of s.

***6.** Write a formula for the square of the sum of three numbers.

7. If the same number is added to each term of the fraction $\frac{e}{f}$, the resulting fraction equals $\frac{g}{h}$. What is the number? Make a numerical illustration and solve it with the help of the formula which is the answer to this question.

8. What is the value of $a^2 - 2ay$ when $a = y - 2$?

9. What is the value of $(x^2 - y^2)a^2 - 4axy$ when $a = \frac{x+y}{x-y}$?

10. A baseball team played g games and lost l of them. What per cent of its games did it win?

11. A basketball team won eight games more than it lost, and it won w games. What per cent of its games did it win?

Represent in algebraic symbols:

12. The diagonal of a square a ft. on each side.

13. The area of the side walls of a room which is l ft. long, w ft. wide, and h ft. high.

14. The length of a diagonal from an upper corner of the room described in the preceding example to the opposite lower corner.

15. The cost of laying a concrete walk w ft. wide and l ft. long, at \$2.50 a sq. yd.

16. The length of the third side of a right triangle of which the hypotenuse is a and another side $\frac{a}{2}$.

Exercise 20. The Solution of Simple Problems

(See pages 45, 52.)

Purpose: To establish scientific habits of thought and to develop the skill required in problem solution.

20, A. Translation or Algebraic Representation

Represent algebraically:

1. The supplement of x degrees; twice the supplement of y degrees; the complement of one third of z degrees.

2. The supplement of a certain angle is $2\frac{1}{2}$ times its complement.

3. One half of a certain number exceeds $\frac{3}{7}$ of it by 4.

4. Three times a certain number is 4 less than $\frac{11}{3}$ of the same number.

5. The total cost of b books at c cents each and of the same number of magazines at 15 cents each.

6. The number of dollars in c cents; the number of cents in $(2x - 3)$ dollars.

7. The annual income on a dollars at 5%; at b %.

8. The selling price of a house bought for d dollars and sold at a profit of p % of the cost.

9. The cost of s suits at d_1 dollars each and of t topcoats at d_2 dollars each, less a dealers' discount of p %.

10. The rate of an airplane moving against a wind of w miles an hour if the plane is making 104 miles an hour through still air.

11. The number of minute spaces passed over in m minutes by the minute hand of a clock; by the hour hand.

12. The part of a book read in an hour by a person who reads the book in h hours.

***13.** Number of pages typed in h hours by a typist who types p pages per hour.

14. The area of a $9' \times 12'$ rug after a border b ft. wide has been added on all four sides.

***15.** The cost of printing 100 copies of a map if the plate costs d dollars, each copy costs c cents, and 12% is added for overhead.

16. The approximate number of bushels in a bin l ft. long, m ft. wide and n ft. high is equal to its volume divided by 1.25.

17. The number of revolutions of a wheel r ft. in radius in going a distance of d ft.

20, B. Algebraic Analysis of Problems

1. In each problem of Exercise 20, C, (1) describe the numbers which are to be represented algebraically; (2) represent them;

(3) state the relation (or relations) upon which you expect to base the equation (or equations).

2. Analyze the problems on pages 46–52 by the method suggested above.

20, C. Problem Solution

(See pages 52 and 168.)

Solve and check each problem below. If you fail on any problem, find the step on which you fail.

1. The sum of the reciprocals of two consecutive even integers is $\frac{13}{84}$. What are the integers?

2. Find two numbers such that 3 times the greater exceeds 4 times the less by 8; and the larger divided by one less than the smaller gives 2.

3. In a certain fraction the denominator is 5 less than the numerator. If the numerator is increased by 6 and the denominator is increased by 1, the value of the fraction is $\frac{9}{4}$. Find the numerator and the denominator.

4. What length and width would you select for a rectangle with a 44 ft. perimeter in order to include an area of 120 sq. ft.?

5. A rectangular box is to be constructed so that its length shall be 5 in. more than its width, and the area of the base shall be 126 sq. in. What should be the dimensions of the base?

6. When a rectangular lot of land was made 2 ft. longer and 2 ft. narrower it was found to contain 6 sq. ft. less area; but when it was made 2 ft. shorter and 3 ft. wider it contained 12 sq. ft. more area. What were the original dimensions? (Make a drawing.)

7. Eighteen years ago A was 7 times as old as B; 12 years hence he will be only twice as old as B is then. What are their ages now? (Arrange the numbers in a table.)

8. A father is 23 years older than his son. How old was the father when the ratio of their ages was $3.\dot{3}$? For two living persons, which numbers remain constant, the sum of their ages; the difference of their ages; the ratio of their ages?

9. A plumber worked on a job from 11.30 A.M. to 8.15 P.M. and was paid for 10 hours. Every hour of overtime counted as $1\frac{1}{3}$ hours. How many hours overtime did he work?

10. A contractor paid 4 plumbers and 5 helpers a total of \$58 a day. They asked for a 10% increase in pay. If he granted the increase and employed 3 plumbers and 6 helpers, they would cost him \$56.10 a day. What was the original daily wage of a plumber? of a helper?

11. Set a selling price for a radio costing \$48 so as to allow $22\frac{1}{2}$% of the selling price for profit and 20% of the selling price for overhead expenses.

12. A boy has $2\frac{1}{4}$ hours for exercise. How long can he ride with his father at 30 miles an hour before getting out to walk back, if he walks $3\frac{3}{4}$ miles an hour?

***13.** Two sprinters start together in a 100-yard race and each runs at a uniform rate. They are 2 yards apart at the end of the first second, and one of them finishes $2\frac{1}{2}$ seconds before the other. What is the speed of each?

***14.** Two gangs of laborers start building walks on both sides of a street 120 ft. long. They start at the same end of the street and one gang on each side of the street. The work is so planned that one gang will do 10 yards a day more than the other and will complete one side and meet the other gang 100 ft. from its starting point. How many days will it take to complete the work? (Use the relation $d = rt$.)

Repeat Exercise 20, A, B, and C from time to time as often as may be necessary. See also the tests on pages 52–54.

Exercise 21. The Use of Number Relations

(See page 61.)

21, A. Understanding

*1. Tell something about how number relations are dealt with in algebra. Mention two ways in which number relations may be expressed.

2. Show by numerical illustration and by an equation that a ratio may be defined as a multiplier.

3. In the equation $a = rb$, when b and r are known, how may a be obtained? When a and r are known, how may b be obtained? When a and b are known, how may r be obtained? Why is this equation often written $a = kb$?

4. In using ratios, what confusion often arises? Suggest two methods by which it may be avoided.

21, B. Algebraic Representation of Number Relations

1. Write a formula stating the fact that b is equal to twice the square of the sum of a and c, less one half the square root of the difference between a and one third c.

2. What is the complement of x degrees? What is $\frac{2}{3}$ the complement of 2 more than x degrees?

3. If an athlete can run 100 yards in b seconds, how many yards can he run in 1 second? in s seconds? (If necessary make use of numerical illustrations.)

*4. If a pupil can typewrite a manuscript in $a + b$ hours, what part of it can he typewrite in 1 hour? If one pupil can type a manuscript in x hours and another in y hours, in how many hours can they both together do it?

5. What is the annual income on a dollars at $b\%$? If this interest is paid in equal monthly payments, how many dollars are there in each payment?

*6. If a man spends a % of his income and gives away b % of it, what fractional part of it is left? If his monthly income is m dollars, how many dollars of his annual income has he left at the end of the year?

21, C. Number Relations in Formulas

1. In the formula $r = \frac{2s}{3t^2}$ if s is unchanged and t is doubled, how is the value of r changed? Find r when $s = 10$ and $t = 4$.

2. In each formula below, tell whether a varies directly as b, as the square of b, as the square root of b, inversely as b, etc.

$$a = \pi b^2 \quad a = \frac{\pi}{b} \quad a = \tfrac{1}{3}(\sqrt{b}) \quad a = \frac{1}{\sqrt{b}} \quad a = kb^3 \quad a = \frac{\sqrt{b}}{k}$$

3. In the formula $a = \frac{b^2}{3c}$, a varies as — (use b): a varies as — (use c). Solve the formula for c and make two statements each beginning, "c varies as —" Solve for b and make two similar statements about b. Find the ratio of a to c, of c to a.

4. In the formula $V = \frac{1}{3}\pi r^2 h$, discuss the relations between V, r, and h according to the plan of the preceding example.

5. In the formula $L = \frac{2 + \frac{1}{a}}{3}$, how does L change as a increases?

21, D. Ratio

1. Corresponding sides of two similar triangles are s and $5s$. What is the ratio of the corresponding sides? Make use of a numerical illustration in order to show what your answer means. Answer the same question for $5n$ and $7\frac{1}{2}n$.

2. The radii of two circles are 3″ and 5″; what is the ratio of the circumferences? of the areas?

3. The diameters of two spheres are d and $d + a$. What is the ratio of their radii? of their circumferences? By how much must the diameter of a sphere be increased in order to double the circumference? What is the difference of the circumferences in the two spheres given above? Use your answer to this question in solving the following problem: If the earth were a perfect sphere 8000 miles in diameter and a steel band were fitted around the Equator, how much would the band need to be lengthened in order to raise it 3 ft. from the earth throughout its entire length?

4. The radii of two spheres are r and $3r$; what is the ratio of their surfaces? of their volumes? ($S = 4\pi r^2$; $V = \frac{4}{3}\pi r^3$.) What change must be made in the radius in order to double the area? the volume?

5. A rectangular solid measures a ft. by b ft. by c ft., and another measures $2a$ ft. by $3b$ ft. by c ft.; what is the ratio of their volumes?

6. A rectangular box is 4 ft. by 6 ft. by $3\frac{1}{2}$ ft., and another is 5 ft. by 3 ft. by $4\frac{1}{2}$ ft.; what is the ratio of their volumes?

7. In a right triangle $a = 10$ ft., $b = 24$ ft., and $c = 26$ ft. Find to three-figure accuracy the ratio of a to b; of a to c; of b to c.

*8. The specific gravity of silver is 10.6. What is the weight of a rectangular block of silver which measures 2 in. by 4 in. by 8 in.? (Water weighs 62.5 lb. per cu. ft.)

*9. The specific gravity of gold is 19.3. Compare the weights of a block of gold and a block of silver of the same size. Express the result as a ratio accurate to the nearest third figure.

21, E. Conversion Ratios

1. Using the ratio 144 for changing a certain number of sq. in. to the corresponding number of sq. ft., do you multiply

or divide by 144? Justify your answer. Write a formula expressing the relation between i, the number of sq. in. in a certain area, and f, the number of sq. ft. in the same area.

2. In a certain right triangle the ratio of a to b is .326. When a is 241 ft., what is b? When b is 200 ft., what is a? Write a formula expressing the relation between a and b.

3. Write a formula showing the relation between f, the number of sq. ft. in a certain area, and y, the number of sq. yds. in the same area. How many sq. ft. are there in 287 sq. yds.? Change 481 sq. ft. to sq. yds. Answer to the nearest third figure. What is the conversion ratio for changing yards to feet? square yards to square feet? cubic yards to cubic feet?

21, F. Verbal Problems

Each problem below is based upon the relation $T = NC$; that is, *total cost* equals *number of articles* times *cost of each article.* See the solutions on pages 45 and 49.

1. Sixty handkerchiefs were bought for a certain sum of money. Half of the money was spent for handkerchiefs at 3 for a dollar and the rest for those at 12 for a dollar. What did the 60 handkerchiefs cost?

2. A man bought a certain number of articles for \$126. By waiting for a sale he could have obtained two more of the articles for the same amount and each article would have cost him \$4 less. How many did he buy? (Compare with Ex. 37, page 49.)

3. A man bought 60 bu. of potatoes for \$108, paying \$1.60 a bushel for part of them and \$1.90 a bushel for the rest. How many bushels did he buy at each price?

4. The Manchester High School bleachers seat 526 persons.

In order to raise \$185 when all the seats are sold, how many must be sold at 50 cents and how many at 25 cents?

Exercise 22. Radicals

(See pages 67–70.)

22, A. Removing Square Factors

(See page 67.)

Leave under the radical sign no factor which is a perfect square. Check by restoring your transformed radical to its original form.

1. $\sqrt{45}$ **2.** $\frac{1}{4}\sqrt{24}$ **3.** $\sqrt{108}$ **4.** $5\sqrt{125}$

5. $\frac{1}{3}\sqrt{27x^2}$ **6.** $x\sqrt{x^2y}$ **7.** $3a\sqrt{12b^3c^4}$ **8.** $\frac{1}{2}\sqrt{48b^5c^3e^4}$

9. Can you simplify in this way $\sqrt{1}$? $\sqrt{2}$? $\sqrt{3}$? $\sqrt{4}$? $\sqrt{5}$?

10. Continue the sequence of the preceding example as far as $\sqrt{100}$.

11. $\frac{x}{4}\sqrt{\frac{x^3}{4}}$ ***12.** $\sqrt{(a^2+2ab+b^2)y}$

***13.** $\frac{1}{2}\sqrt{a^2y-2aby+b^2y}$ ***14.** $-a\sqrt{3a^2(a-1)^3}$

***15.** $a\sqrt{(a+b)^2}$ ***16.** $\sqrt{36(a^2-x^2)(a+x)}$

17. $\sqrt{-4}$ **18.** $\sqrt{-18}$ **19.** $\sqrt{9-18a}$ ***20.** $\sqrt{4-8\sqrt{5}}$

***21.** $\sqrt{8-4\sqrt{2}}$ **22.** $\sqrt{288}$ **23.** $\sqrt{1200+50}$

22, B. Addition of Radicals

(See page 67.)

Unite if possible:

1.

$3\sqrt{2}$	$\sqrt{2}$	$x\sqrt{3}$	$x\sqrt{5}$	$a\sqrt{5}$	$a\sqrt{3}$	$3\sqrt{a}$
$5\sqrt{2}$	$-3\sqrt{2}$	$v\sqrt{3}$	$-\sqrt{5}$	$a\sqrt{3}$	$a\sqrt{3}$	$-4\sqrt{a}$

*2. $\sqrt{44}-\sqrt{11}$ *3. $\sqrt{8}-\sqrt{18}$ *4. $\sqrt{20}-\sqrt{45}$

*5. $\sqrt{50}-\sqrt{72}$ *6. $3\sqrt{27}-\sqrt{12}-3\sqrt{18}$

*7. $4\sqrt{3}+2\sqrt{12}+\sqrt{75}$

*8. $\frac{1}{2}\sqrt{12}-\sqrt{50}+\frac{1}{3}\sqrt{48}-\sqrt{18}$

*9. $a\sqrt{a^3b}+\sqrt{ab^5}-\sqrt{4a^3b^3}$

*10. $\sqrt{(a-b)^2y}+\sqrt{(a+b)^2y}-\sqrt{a^2y}+\sqrt{(1-a)^2y}-\sqrt{y}$

*11. $.04\sqrt{98}-.05\sqrt{72}+\frac{1}{5}\sqrt{50}$

*12. $\sqrt{3^2\cdot 2}-\sqrt{4^2\cdot 3}+\frac{1}{2}\sqrt{8^2\cdot 2}-\frac{3}{5}\sqrt{5^2\cdot 2^2\cdot 3}$

*13. $(\sqrt{a+b}+\sqrt{a-b})+(\sqrt{a+b}-\sqrt{a-b})$

22, C. Multiplication

(See page 68.)

Multiply:

1. $\sqrt{2}\times\sqrt{3}$ 2. $2\sqrt{3}\times 3\sqrt{2}$ 3. $\frac{1}{2}\sqrt{6}\times\frac{3}{4}\sqrt{3}$

4. $a\sqrt{b}\times\sqrt{ab}$ 5. $(\sqrt{20}+\sqrt{80}-\sqrt{45})\sqrt{5}$

6. $(\sqrt{8}-2\sqrt{12}+\sqrt{20})(\sqrt{6})$ 7. $(4-\sqrt{7})^2$

8. $(4+\sqrt{7})^2$ 9. $(4-\sqrt{7})(4+\sqrt{7})$ 10. $(2\sqrt{3}-3\sqrt{2})^2$

11. $(2\sqrt{3}+3\sqrt{2})^2$ 12. $(2\sqrt{3}-3\sqrt{2})(2\sqrt{3}+3\sqrt{2})$

13. $(\sqrt{a}-\sqrt{b})^2$ *14. $(\frac{1}{2}\sqrt{8a}-\frac{3}{4}\sqrt{32b})(\frac{1}{2}\sqrt{2a}-\frac{1}{4}\sqrt{b})$

15. $(\sqrt{x-2}-\sqrt{x+2})(\sqrt{x-2}+\sqrt{x+2})$

16. $(\sqrt{10}+\sqrt{19})(\sqrt{10}-\sqrt{19})$

17. $(\sqrt{x+2}+\sqrt{x})(\sqrt{x+2})$ 18. $(\sqrt{2a}-\sqrt{3})(\sqrt{2a}+\sqrt{3})$

19. $(a\sqrt{3}-2)(a\sqrt{3}-2)$ 20. $(\sqrt{5a}+6)(2\sqrt{5a}+1)$

21. $(\sqrt{11}-2\sqrt{2})(2\sqrt{11}+\sqrt{2})$ 22. $(\sqrt{8}-9)(3\sqrt{8}+1)$

23. $(a\sqrt{b}-2)(a\sqrt{b}+2)$ **24.** $(\sqrt{x+2}-5)(\sqrt{3x+6}+5)$

25. $(\sqrt{x-3}-7)^2$ **26.** $(\sqrt{2x-14}-3)(\sqrt{x-7}+2)$

***27.** $(\sqrt{a-b}-c)(\sqrt{2a-2b}+c)$

***28.** $(-b+\sqrt{b^2-4ac})(-b-\sqrt{b^2-4ac})$

29. $(\sqrt{a-3}-\sqrt{a+3})^2$

30. Multiply each of the following binomials by a binomial which will give a rational product:

$\sqrt{3}-5$ $\sqrt{a}-\sqrt{b}$ $3\sqrt{5}-6\sqrt{7}$ $\sqrt{x-3}+4$

31. $(\sqrt{3x-5}-7)(\sqrt{3x-5}-8)$ ***32.** $[(a-b)-\sqrt{c}]^2$

***33.** $[(a+b)\sqrt{r}]\,[(a-b)\sqrt{r}]$

22, D. Removal of Fractions

(See page 68.)

Leave no fraction under the radical: check by restoring each transformed radical to its original form.

1. $\sqrt{\frac{1}{2}}$ **2.** $\sqrt{\frac{1}{3}}$ **3.** $\sqrt{\frac{2}{3}}$ **4.** $\sqrt{\frac{3}{4}}$

5. $\sqrt{\frac{3}{5}}$ **6.** $\sqrt{\frac{7}{8}}$ **7.** $\frac{1}{2}\sqrt{\frac{2}{3}}$ **8.** $5\sqrt{\frac{4}{5}}$

9. $\sqrt{\dfrac{1}{x}}$ **10.** $\sqrt{\dfrac{1}{2x}}$ ***11.** $\sqrt{\dfrac{2}{x^3}}$ ***12.** $3a\sqrt{\dfrac{5}{27a^2}}$

***13.** $\sqrt{2^2-(\frac{3}{5})^2}$ ***14.** $\sqrt{6^2+(\frac{4}{3})^2}$ ***15.** $\sqrt{x^2-\left(\dfrac{x}{3}\right)^2}$

***16.** $\sqrt{\left(\dfrac{a+4}{4}\right)^2-a}$ **17.** $\sqrt{\dfrac{3x}{5y}}$ ***18.** $4\sqrt{\dfrac{x^2y}{8}}$

19. $\sqrt{1\frac{2}{5}}$ ***20.** $\sqrt{x^2+\dfrac{x^2}{4}}$ ***21.** $\sqrt{2m^2+\left(\dfrac{m}{2}\right)^2}$

22. $5\sqrt{\frac{27}{5}}$ **23.** $6\sqrt{\frac{25}{18}}$ **24.** $\frac{3}{4}\sqrt{\frac{12}{25}}$ ***25.** $2\sqrt{28}+\sqrt{\frac{1}{7}}$

***26.** $\sqrt{\frac{1}{5}}+\sqrt{5}$ ***27.** $\sqrt{\frac{1}{3}}-\sqrt{3}$ ***28.** $\sqrt{4a^3}-\sqrt{\frac{4}{a}}$

29. Is $\sqrt{-\frac{c}{a}}\equiv\frac{\sqrt{-ac}}{a}$? **30.** Is $\sqrt{\frac{1}{5}+3}\equiv 4\sqrt{\frac{1}{5}}$?

***31.** $\sqrt{\frac{1}{2}}+\sqrt{\frac{2}{9}}+2\sqrt{32}-\sqrt{\frac{8}{25}}$

***32.** $2\sqrt{20}+\frac{1}{3}\sqrt{45}-5\sqrt{\frac{1}{5}}+2\sqrt{80}$

***33.** $7\sqrt{\frac{1}{5}}-5\sqrt{\frac{1}{20}}-\frac{1}{5}\sqrt{20}$

22, E. Rationalization

1. Give a rationalizing factor for each denominator in Exercises 2–26.

Express with rational denominators:

2. $\frac{1}{\sqrt{2}}$ **3.** $\frac{1}{\sqrt{3}}$ **4.** $\frac{3}{\sqrt{3}}$ **5.** $\frac{6}{\sqrt{8}}$ **6.** $\frac{5\sqrt{15}}{2\sqrt{5}}$

7. $\frac{4\sqrt{2}}{3\sqrt{8}}$ **8.** $\frac{\sqrt{4}}{\sqrt{3}}$ **9.** $\frac{5}{\sqrt{m+n}}$ **10.** $\frac{\sqrt{s}}{\sqrt{s-t}}$

11. $\frac{\sqrt{s}-\sqrt{t}}{\sqrt{s-t}}$ **12.** $\frac{\sqrt{s}+\sqrt{t}}{\sqrt{st}}$ **13.** $\frac{\sqrt{s}}{\sqrt{s}-\sqrt{t}}$

14. $12\div 6\sqrt{3}$ **15.** $18\div 3\sqrt{6}$ **16.** $75\div 5\sqrt{27}$

17. $\frac{2}{2-\sqrt{3}}$ **18.** $\frac{3\sqrt{5}-4}{2-\sqrt{5}}$ **19.** $\frac{3\sqrt{2}}{3-\sqrt{2}}$

20. $\frac{\sqrt{10}}{\sqrt{2}+\sqrt{5}}$ **21.** $\frac{\sqrt{2}+\sqrt{3}}{\sqrt{3}-\sqrt{2}}$ **22.** $\frac{6\sqrt{2}+5\sqrt{7}}{5\sqrt{2}-6\sqrt{7}}$

***23.** $\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$ ***24.** $\frac{\sqrt{a}+5\sqrt{b}}{\sqrt{a}+2\sqrt{b}}$ ***25.** $\frac{\sqrt{a^2-3}}{\sqrt{a^2-3}-3}$

*26. $\dfrac{\sqrt{2}}{\sqrt{2a^2+2}-4}$ *27. $\dfrac{6}{\sqrt{3x-5}-1}$

Divide, and express the quotients as fractions with rational denominators:

28. $(\sqrt{5}-3\sqrt{7}) \div (\sqrt{5}+3\sqrt{7})$

29. $(3\sqrt{2}-\sqrt{5}) \div (\sqrt{2}-2\sqrt{5})$

*30. $\dfrac{(\sqrt{2}+\sqrt{3})+\sqrt{5}}{(\sqrt{2}+\sqrt{3})-\sqrt{5}}$ *Plan of work:* Multiply numerator and denominator by $[(\sqrt{2}+\sqrt{3})+\sqrt{5}]$, simplify and multiply by $\sqrt{6}$.

22, F. Finding Numerical Values

(See page 68.)

Simplify each of the radical expressions below and find its numerical value to the nearest third figure:

1. $\sqrt{12}$ 2. $\sqrt{18}$ 3. $\sqrt{50}$ 4. $\sqrt{20}$ 5. $\sqrt{8}$ 6. $\sqrt{\frac{1}{2}}$

7. $\sqrt{\frac{2}{3}}$ 8. $\sqrt{\frac{1}{5}}$ 9. $3\sqrt{\frac{1}{6}}$ 10. $\dfrac{1}{\sqrt{3}}$ 11. $\dfrac{2}{\sqrt{5}}$ 12. $\dfrac{\sqrt{2}}{\sqrt{5}}$

13. $\dfrac{\sqrt{15}}{\sqrt{3}}$ 14. $\dfrac{\sqrt{18}}{\sqrt{2}}$ 15. $\sqrt{3}\times\sqrt{15}$ 16. $\sqrt{2}\times 4\sqrt{3}$

17. $\frac{2}{3}\sqrt{8}\times\sqrt{2}$ 18. $(\sqrt{3})^2$ 19. $(2\sqrt{5})^2$

20. $(\sqrt{2}-\sqrt{5})(\sqrt{2}+\sqrt{5})$ 21. $\sqrt{4+9}$ 22. $\sqrt{4-1}$

*23. $\sqrt{\frac{1}{4}}-\sqrt{\frac{1}{16}}+\sqrt{98}-2\sqrt{27}$ *24. $2\sqrt{50}-\sqrt{4}-6\sqrt{\frac{3}{2}}$

*25. $\sqrt{6}-\sqrt{\frac{2}{3}}+4\sqrt{6}-\sqrt{\frac{1}{6}}$ *26. $3\sqrt{10}\,(\sqrt{2}-2\sqrt{5})$

*27. $\dfrac{3}{\sqrt{8}}$ *28. $(2\sqrt{3}+3\sqrt{5}) \div \sqrt{15}$ 29. $\dfrac{5+\sqrt{3}}{\sqrt{3}}$

30. $(5\sqrt{6}-7\sqrt{8})(5\sqrt{6}+7\sqrt{8})$ *31. $\sqrt{5}(5\sqrt{3}+2\sqrt{15})$

***32.** $2\sqrt{3}(4\sqrt{3}+\sqrt{7})(-2\sqrt{7})(\sqrt{3}+3\sqrt{7})$

***33.** $6\sqrt{68}+\sqrt{\frac{17}{4}}-\frac{1}{2}\sqrt{7}$

***34.** $(2\sqrt{3}-3\sqrt{5})\div(2\sqrt{3}+3\sqrt{5})$

***35.** Find numerical values of the numerical expressions of Exercises **22,** D and E.

22, G. Radicals. Miscellaneous

(See page 70.)

1. When is a radical said in this course to be in "simplest form"? Is this form the most convenient for *all* purposes? Explain or illustrate your answer. Why does the transformation of radicals receive less emphasis now than formerly?

"Simplify" the following radical expressions, or perform the indicated operations and "simplify" the results:

2. $\sqrt{20}$ **3.** $\sqrt{75}$ **4.** $\sqrt{128}$ **5.** $\sqrt{99}$ **6.** $\sqrt[3]{162}$

7. $\sqrt[4]{405}$ **8.** $\sqrt{\frac{1}{2}}$ **9.** $\sqrt{\frac{2}{3}}$ **10.** $\sqrt{\frac{3}{5}}$ **11.** $\sqrt[3]{\frac{3}{4}}$

12. $\sqrt{\frac{5}{8}}$ **13.** $\sqrt[3]{\frac{1}{2}}$ **14.** $\sqrt{\frac{1}{3}a}$ ***15.** $\sqrt{\dfrac{2}{a^3}}$ ***16.** $\sqrt[3]{\dfrac{1}{m^2}}$

***17.** $\sqrt[3]{\dfrac{1}{ab^3}}$ **18.** $\sqrt[3]{\frac{5}{27}}$ **19.** $\sqrt[3]{-\frac{4}{9}}$ ***20.** $\sqrt{3^2-(\frac{2}{3})^2}$

***21.** $\sqrt{6^2+(\frac{4}{5})^2}$ ***22.** $\sqrt{a^2-\left(\dfrac{a}{3}\right)^2}$ ***23.** $\sqrt{\left(\dfrac{x+4}{4}\right)^2+x}$

***24.** $\sqrt{48}-\sqrt{12}+\sqrt{75}$ ***25.** $2\sqrt{18}-5\sqrt{50}+3\sqrt{98}$

***26.** $2\sqrt{343}+5\sqrt{175}-17\sqrt{112}$

***27.** $3\sqrt[3]{2}-2\sqrt[3]{16}+7\sqrt[3]{54}$

***28.** $\sqrt[3]{375}+2\sqrt[3]{24}-\sqrt[3]{648}$ ***29.** $8\sqrt{\frac{2}{3}}+2\sqrt{\frac{3}{2}}-5\sqrt{\frac{1}{6}}$

*30. $a\sqrt{b^3} + \sqrt{a^2b^3} - 2\,b\sqrt{a^2b}$ *31. $\sqrt[3]{x^2y} + \sqrt[3]{x^5y} - \sqrt[3]{x^2y^4}$

*32. $\sqrt[4]{3} + \sqrt[4]{243}$ 33. $5\sqrt[3]{108} - \sqrt[3]{32} + \sqrt[3]{4}$

*34. $\sqrt{\dfrac{2\,a}{b}} - \sqrt{\dfrac{2\,b}{a}} + \sqrt{\dfrac{a}{2\,b}}$ 35. $\dfrac{\sqrt{6}}{\sqrt{2}}$ 36. $\dfrac{\sqrt{12\,a}}{\sqrt{3\,a}}$

37. $\dfrac{\sqrt{\frac{1}{2}}}{\sqrt{\frac{1}{3}}}$ 38. $\dfrac{2}{10 - \sqrt{6}}$ 39. $\dfrac{3}{5 + \sqrt{2}}$

40. $\dfrac{7}{\sqrt{2} + \sqrt{3}}$ 41. $\dfrac{\sqrt{2}}{\sqrt{3} - \sqrt{7}}$

*42. $\dfrac{\sqrt{a} + \sqrt{b}}{\sqrt{a} - \sqrt{b}}$ 43. $(5 + \sqrt{6}) \div (5 - \sqrt{6})$

Solve for each letter in turn:

44. $r = \sqrt{\dfrac{V}{\pi h}}$ 45. $r = \sqrt{\dfrac{A}{4\,\pi}}$

46. Extract a monomial factor from $\sqrt{4 - 8\sqrt{a}}$; from $\sqrt{9 - 3\sqrt{72}}$.

*47. $\dfrac{a^2\sqrt{a^2b^3c}}{a\sqrt{ab^2c}}$ 48. $\dfrac{(m^2 - n^2)\sqrt{abc}}{(m + n)\sqrt{ac}}$

*49. $\sqrt{a^3c} - \sqrt{ac^3} + \sqrt{\dfrac{a}{c}}$

50. $\dfrac{-b + \sqrt{b^2 - 4\,ac}}{2a} + \dfrac{-b - \sqrt{b^2 - 4\,ac}}{2a}$

51. $\left(\dfrac{-b + \sqrt{b^2 - 4\,ac}}{2a}\right)\left(\dfrac{-b - \sqrt{b^2 - 4\,ac}}{2a}\right)$

*52. $\sqrt{3\,a^2 - 6\,a^3b}$ *53. $\sqrt[3]{24} - \sqrt[3]{81} + \sqrt[3]{54}$

*54. $\sqrt{2x} + \sqrt[3]{16x} + \sqrt[3]{2x} + \sqrt{50x}$

*55. $\sqrt[3]{a^2x^4} + \sqrt[3]{125a^2x} - \sqrt[3]{27a^5x}$

56. $\sqrt{\frac{2}{3}} \times \dfrac{3}{\sqrt{2}}$

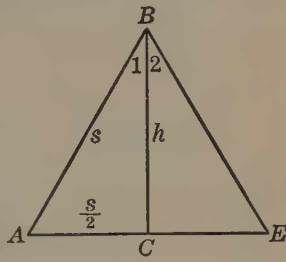
22, H. Formulas Involving Radicals

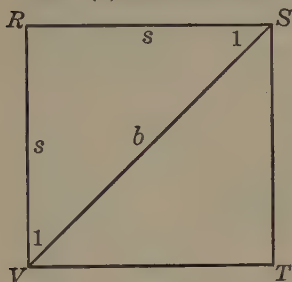
(See page 70.)

Solve each formula for the letter indicated, and then find the missing numerical values. Notice that "simplification" of the radicals does not always simplify the numerical computation. Notice also that the constants have been so selected that the square roots are exact.

1. $V = c^2rs$ for c. $V = 400$, $r = 153$, $s = 17$, $c = ?$
2. $s = \frac{1}{2}gt^2$ for t. $s = 326.025$, $g = 32.2$, $t = ?$
3. $V = \frac{1}{3}\pi r^2h$ for r. $V = 942\frac{6}{7}$, $\pi = \frac{22}{7}$, $h = 16$, $r = ?$
4. $s = \dfrac{bd^2}{6}$ for d. $s = 262\frac{1}{2}$, $b = 28$, $d = ?$
5. $K = \frac{1}{2}mv^2$ for v.
6. $S = 4\pi r^2$ for r. $S = 24.64$, $\pi = \frac{22}{7}$, $r = ?$
7. $a^2 + b^2 = c^2$ for b and for a. $a = 4.45$, $c = 13.30$, $b = ?$
   $b = 8.81$, $c = 9.50$, $a = ?$
8. $\dfrac{a}{b} = \dfrac{b}{c}$ for b. $a = 18.40$, $b = 4.60$, $c = ?$
9. $K = \dfrac{Wv^2}{2g}$ for v. $K = 1012\frac{1}{2}$, $W = 200$, $g = 32$, $v = ?$
10. $V = \frac{4}{3}\pi r^3$ for r.
11. Find to three-figure accuracy the side of an equilateral triangle of which the altitude is 3.56″.

12. *The* 30° 60° *right triangle.* ABE is an equilateral triangle. BC bisects angle B and is perpendicular to AE; therefore $B_1 = 30°$ and the angles at C are right angles. $AC = \frac{1}{2}AE$. The triangle ABC is a 30° 60° right triangle. Express: (1) h in terms of s. (2) s in terms of h. (3) The area (A) in terms of s. (4) s in terms of A. (5) A in terms of h. (6) h in terms of A.





13. *The* 45° *right triangle.* $RSTV$ is a square. $V_1 = S_1 = 45°$. The triangle VRS is a 45° right triangle, or an isosceles right triangle. Express (1) b in terms of s, (2) s in terms of b, (3) the area S in terms of b, (4) the area in terms of s.

Exercise 23. Parentheses

(See page 71.)

23, A. A Matching Exercise

Indicate which of the following pairs of expressions are equivalent and which are not equivalent:

1. $5x^2 \qquad (5x)^2$ **2.** $5(a+b)^2 \qquad 5a^2+5b^2$

3. $\sqrt{x^2-2xy+y^2} \qquad x-y$ **4.** $\sqrt{4}+9 \qquad \sqrt{4+9}$

5. $5-\overline{a+5} \qquad 5-(a+5)$ **6.** $\sqrt{\dfrac{x}{y}} \qquad \dfrac{\sqrt{x}}{\sqrt{y}}$

7. $-\dfrac{-x}{y} \qquad \dfrac{x}{y}$ **8.** $-(a-b) \qquad (b-a)$

9. $-(x-5) \qquad (x+5)$ **10.** $\dfrac{x-y}{3} \qquad -\dfrac{y-x}{3}$

11. $-\dfrac{a+2}{5}$ $\qquad \dfrac{2-a}{5}$

12. $\dfrac{2}{\frac{5}{3}}$, $\qquad \dfrac{\frac{2}{5}}{3}$

13. $[(a+b)+(x+y)]^2$ $\qquad (a+b)^2+2(a+b)(x+y)+(x+y)^2$

23, B. The Uses of Parentheses

(See page 71.)

1. In your opinion what purpose do parentheses serve?

Perform the operations indicated: express results in simple form:

2. $10+(5-2)$
3. $12-(6-4)$
4. $15-2(7-4)$
5. $a+(b-a)$
6. $b-(c-b)$
7. $5-\overline{3x-1}$
8. $3x-\overline{x-3}$
9. $\dfrac{3x-3y}{3}+\sqrt{(x-y)^2}-\dfrac{x^2-2xy+y^2}{x-y}-(x-y)$
10. $9-[4-(3-1)]$
11. $10+[4+3-4(9-4)]$
12. $[(a+b)+7][(a+b)-2]$
13. $[(a+b)-4][(a+b)+4]$
14. $[(a+b)+c][(a+b)+c]$
15. $[(a-b)+(c-e)]^2$
16. $a(-1)^2$
17. $(-1)(-1)(-1)^2$
18. $\dfrac{4a}{b-a}\times\dfrac{-1}{-1}$
19. $\left(\dfrac{-3x}{4y}\right)^2$
20. $\left(\dfrac{-2x}{5y}\right)^3$
21. $\dfrac{x^2-y^2}{y-x}$
22. $(a-2b)^2$
23. $[(a+c)-3]^2$
24. $(x+3)^2-(x-3)^2$
25. $[(x+5)-(x-5)]^2$
26. $(a+b)x+(a+b)y$
27. $(a+b)x+(b-c)x$
28. $(b+c)x+x$
29. $(r-s-t)x-x$
30. $a\sqrt{3}-b\sqrt{3}-\sqrt{3}$
31. $\dfrac{x-2}{3}-\dfrac{2-x}{3}$
32. $\dfrac{2a-b}{5}-\dfrac{b-2a}{5}$

33. $\dfrac{2x-3}{7}-\dfrac{3-2x}{7}$ **34.** $\dfrac{5}{a-b}-\dfrac{5}{b-a}$

35. $5+3(2-7)-3-(3-4)$

36. $(21-3\times 4)(17-5\times 2)-3\times 7$

37. $\sqrt{x^2+10x+25}$ **38.** $\sqrt{x^2-12x+36}$

39. $\sqrt{4x^2-4y^2}$ **40.** $\dfrac{\sqrt{8}}{\sqrt{2}}$ **41.** $\dfrac{\sqrt{27}}{\sqrt{3}}$

42. $\dfrac{\sqrt{8}+\sqrt{2}}{\sqrt{2}}$ **43.** $\dfrac{\sqrt{8+2}}{\sqrt{2}}$ **44.** $(2-\sqrt{7})(2+\sqrt{7})$

45. $(2-\sqrt{5})\div(3-\sqrt{5})$ **46.** $(a-\sqrt{3})\div(a-\sqrt{6})$

47. $(2-\sqrt{6})\div(3-\sqrt{6})$ ***48.** $[3-(x+4)][3+(x+4)]$

***49.** $[s-x-y][s+x+y]$ ***50.** $5a-[1.7-(2a-4.8)]$

51. $1.2\{4-(.3-x)-x\}$ **52.** $5x-\{-x-(x-2)\}$

***53.** $a-[25b+(a-16b+9-10b)]$

54. $\dfrac{a^3+b^3}{a+b}$ **55.** $\dfrac{a^3+b^3}{a-b}$

23, C. Removing Quantities from Parentheses and Inserting Quantities into Parentheses

(See page 71.)

Supply the missing quantities. Check your work.

1. $4(a-b)\equiv(\quad ?\quad)$ **2.** $4-(a-b)\equiv(\quad ?\quad)$

3. $4(a-b)^2\equiv(\quad ?\quad)$ **4.** $4\sqrt{a-b}\equiv\sqrt{\quad ?\quad}$

5. $4\left(\dfrac{a-b}{3}\right)\equiv\dfrac{?}{3}$ **6.** $4\left(\dfrac{3}{a-b}\right)\equiv\dfrac{?}{a-b}$

7. $4 - \dfrac{3}{a-b} \equiv \dfrac{?}{a-b}$

8. $3 \equiv \dfrac{?}{a+2b}$

9. $3 \equiv \dfrac{?}{a-b}$

10. $a + \dfrac{5}{2a-b} \equiv \dfrac{?}{2a-b}$

11. $a^2 - \dfrac{a^2}{1-b^2} \equiv \dfrac{?}{?}$

12. $4\frac{3}{7} \equiv \dfrac{?}{7}$

13. $2ab^2 - 6ac - 12 \equiv 2(\quad ? \quad)$

14. $4a^2b - 8ab^3 - 12b^2 \equiv 4b^2(\quad ? \quad)$

15. $2\pi r^2 + 2\pi rh \equiv ?(\quad)$

16. $180n - 360 \equiv ?(\quad ? \quad)$

17. $x^2 - 10xy + 25y^2 \equiv (\quad ? \quad)^2$

18. $a + b - 2c \equiv \frac{1}{2}(\quad ? \quad)$

19. $a^2 - 2b^2 - 3c \equiv \frac{1}{3}(\quad ? \quad)$

20. $b - a \equiv -(\quad ? \quad)$

21. $b - a - c \equiv b - (\quad ? \quad)$

22. $-7a + 8b - c \equiv -(\quad ? \quad)$

23. $2b^2 - 4a^2 - 6c^2 \equiv -2(\quad ? \quad)$

24. $(3a-2b)x - (3a-2b)y \equiv (3a-2b)(\quad ? \quad)$

25. $a(x-c) - bx + bc \equiv (x-c)(\quad ? \quad)$

26. $2(x-b) - 3ax + 3ab \equiv (x-b)(\quad ? \quad)$

27. $4(ab^2)^2 \equiv (\quad ? \quad)^2$

28. $a^2(bc^3)^2 \equiv (\quad ? \quad)^2$

29. $16a^2(bc)^2 \equiv (\quad ? \quad)^2$

30. $9(x+y)^2 \equiv (\quad ? \quad)^2$

*31. $27(a-b)^3 \equiv (\quad ? \quad)^3$

32. $2a \equiv \sqrt{?}$

33. $5\sqrt{a} \equiv \sqrt{?}$

34. $\frac{1}{2}\sqrt{a} \equiv \sqrt{?}$

35. $2 \equiv \sqrt[3]{?}$

36. $4\sqrt[3]{a} \equiv \sqrt[3]{?}$

37. $3\left(\dfrac{x-3}{7}\right) \equiv \dfrac{(?)}{?}$

*38. $5 \equiv \dfrac{?}{\sqrt{2}-\sqrt{5}}$

EXERCISES FOR CHAPTER II

TRIGONOMETRY

Exercise 24. Finding Trigonometric Ratios Without the Use of Tables

(See page 82.)

Aim: To make sure that you understand what trigonometric ratios are.

1. Define the sine of an acute angle; the cosine; the tangent.

2. In a right triangle ABC, $a = 12$, $b = 16$, and $c = 20$. Find the sine, cosine, and tangent of A and of B. Express results as common fractions.

3. Find three functions of A and of B in the right triangle in which $a = 21$ and $b = 20$.

4. In a right triangle $\sin A = \frac{24}{25}$. Find two other functions of A and B.

5. In a right triangle $\tan A = \frac{5}{12}$. Find two other functions of A and three functions of B.

6. Find three functions of an acute angle of an isosceles right triangle. Express results in radical form.

7. In a right triangle, the hypotenuse is twice the shortest side. Find three functions of the smaller acute angle.

8. Find the sine, cosine, and tangent of 30°, 60°, and 45°. Express results to four-figure accuracy.

9. Show that the sine of an acute angle is always less than 1.

Exercise 25. The Use of Tables of Trigonometric Functions (See pages 84–85.)

25, A. Direct

Find in the table, using pencil only when necessary:

1. sin 37.8°; cos 50.9°; tan 88.2°; cos 31.6°
2. sin 22° 10′; cos 43° 20′; tan 65° 30′; cos 76° 40′;
3. sin 29.31°; 29.32°; 29.33°; 29.34°; 29.35°; 29.39°.
4. tan 2.37°; 21.08°; 56.99°; 70.02°; 68°51′; 79°6′.
5. cos 14.11°; 9° 34′; 25.22°; 61.09°; 6°27′; 70°2′.

25, B. Indirect

Supply the missing numbers:

1. sin ( ) = 0.2011; tan ( ) = 6.107; cos ( ) = 0.5764.

Find to the nearest tenth of a degree:

2. tan ( ) = 0.10578; cos ( ) = 0.6000; cos ( ) = 0.9868.

Find to the nearest hundredth of a degree:

3. sin ( ) = 0.5555; tan ( ) = 0.5555; cos ( ) = 0.5555.
4. sin ( ) = 0.2222; tan ( ) = 0.2222; cos ( ) = 2.2222.
5. In examples 3 and 4, express your results in degrees and minutes to the nearest minute.
6. In a right triangle in which $a = 3.000$ and $b = 4.000$, find A and B to the nearest hundredth of a degree.
7. Supply the missing numbers or tell why you cannot do so: sin ( ) = 7.269; cos ( ) = 7.269; tan ( ) = 7.269.
8. Show with the help of the tables that $\sin 24° + \sin 32° \neq \sin (24° + 32°)$. In general $\sin A + \sin B \neq \sin (A + B)$.
9. Find in the tables the limiting values of the sine of an acute angle, of the cosine, of the tangent.

Exercise 26. Solution of Right Triangles

(See page 87.)

Solve the following right triangles. In each example work to the degree of accuracy indicated by the least accurate datum. Check your results.

	A	B	a	b	c
1.		30.6°			4.14″
2.				8.20″	17.7″
3.	36° 42′		136.0′		
4.	41.3°			27.0′	
5.			15.7′		22.6′
*6.			12.71′		17.38′
*7.		45.16°	37.22′		
*8.	56°16′		37.21′		

EXERCISES FOR CHAPTER III

Object: To test your skill with exponents; and, more important still, to test your ability to understand the general statement of a law and to apply the law.

Exercise 27. Exponents (Positive and Integral)

27, A. (See page 101.)

***1.** State in symbols the four laws of exponents. (State them in words.)

2. What is the importance of literal exponents? Can you illustrate?

3. What do we mean by the "degree of a term"? Illustrate.

Justify your answer to each question in Examples 4–9:

4. Is $x^3 \cdot x^4 \equiv x^{12}$ or x^7? **5.** Is $2^3 \cdot 2^2 \equiv 2^5$ or 4^5?

6. Is $\left(\frac{x}{y}\right)^4 \equiv \frac{x^4}{y}$ or $\frac{x^4}{y^4}$?

7. Is $2(x^2y^3)^2 \equiv 2^2\,x^4y^6$ or $2\,x^4y^6$?

8. Is $\sqrt[3]{8\,x^8} \equiv 2\,x^2$ or $2\,x^{\frac{8}{3}}$?

9. Is $2^3 \cdot 4^2 \equiv 8^5$?

10. Express with one exponent:

$2^2 \cdot 2^3 \qquad 2 \cdot 2^2 \qquad 2^4 \cdot 2^8 \qquad 2^3(-2)^3 \qquad 2^3(-2^3)$

Perform the operations indicated:

11. $\frac{a^2x^5y^3}{b^2c^3} \cdot \frac{a^5x^2y}{bc^2} \div \frac{axy}{bc^3}$

12. $\frac{m^2np^3}{rt^4} \cdot \frac{mn^2t}{r^3p^2} \div \frac{m^3n^2p}{r^4t^3}$

13. $\frac{a^2b^3c}{x^2y^3z} \times \frac{x^2y^3z}{a^2b^3c}$

14. $x^3 \cdot x^2 \qquad x^4 \cdot x \qquad x \cdot x^3 \qquad x(-x^2) \qquad x(-x)^2$

15. $x^a \cdot x^b \qquad x^a \cdot x^b \cdot x^c \qquad x^a \cdot x \qquad x^{n-1} \cdot x \qquad x^{n+1} \cdot x$

16. $x^a \cdot x^2 \qquad x^{2a} \cdot x^3 \qquad x^{2a} \cdot x^{3b} \qquad x^{4b} \cdot x \qquad x^a \cdot x^{a+1}$

17. $x^3 \cdot x^3 \qquad (x^2)^3 \qquad (-x^3)^2 \qquad (2\,x^2)^2 \qquad (-3\,x^3)^3$

18. $(x^a)^b \qquad (x^{2a})^b \qquad (x^{2a})^z \qquad (x^{n+1})^2 \qquad (x^{2n-3})^{2n-3}$

19. $\sqrt{x^6} \qquad \sqrt[3]{x^6} \qquad \sqrt[4]{x^8y^2} \qquad \sqrt{x^{2a}} \qquad \sqrt[a]{x^{6a}}$

20. $\frac{x^7}{x^4} \qquad \frac{x^4}{x^4} \qquad \frac{x^5}{-x} \qquad \frac{x^a}{x} \qquad \frac{a^x}{a^y} \qquad \frac{x^{n+1}}{x} \qquad \frac{-x^{2a-1} \cdot x^{3a-2}}{-x^{a+1}}$

21. Multiply:

$$\begin{array}{c} ab^2c^3 \\ \underline{a^mb^{2m}c} \end{array} \qquad \begin{array}{c} -5\,a^rb^s \\ \underline{-3\,a^{r-2}b^2} \end{array} \qquad \begin{array}{c} -3\,ay^{n-2} \\ \underline{2\,a^2y^{n-1}} \end{array}$$

*22. $\frac{2\,x^2(a-b)^3}{-x(a-b)^2}$

23. $\sqrt{a^3b^{n-1} - 3\,a^4b^{2n-1} + 2\,a^2b^{3n-2}}$

24. Are the following expressions identical?

$a^6 - b^6 \qquad (a^2)^3 - (b^2)^3 \qquad (a^3)^2 - (b^3)^2$

27, B

Multiply. Check by division. When you can, write the results by inspection.

1. $a(b-a-1)$ $a^2b(a^2-b^2-c)$ $(a-b)(a+b)$

2. $(a-b)^2$ $(a+b)^2$ $(x+5)^2$ $(x-7)^2$

3. $(x-y)(x^2-2\,xy+y^2)$ $(x+y)(x^2-2\,xy+y^2)$

4. $(x-y)(x-2\,xy-y)$

5. $(a+b+c)^2$ $(x-y-z)^2$ $(a-b+c)^2$

6. $a(x-y)^3$. Check by numerical substitution.

7. $b(x-y)^3$ $2(x-8)^3$ $3(x-2\,y)^3$

8. $(2-3)(2+3)$ $(2+3)^2$ $(2-3)^2$

9. $(x^a+y^c)^2$ $(x^a+y^c)(x^a-y^c)$ $(x^a-y^c)^2$

10. Does $ay(a+3)-a(a^2-10)=y+2$, if $y=a-3$?

11. If $x=4-y$ and $x^2-xy+y^2=7$, substitute the first equation in the second and solve the resulting equation for y.

Substitute and solve for y as in the preceding example:

12. $x=y-1$ and $3\,x^2+xy+3\,y^2=17$

13. $x=7-2\,y$ and $x^2+6\,xy-3\,y^2=2\,x+6$

14. $x=\dfrac{2}{y}$ and $4\,x^2+y^2-17=0$

15. $x=\dfrac{8-y}{2}$ and $8+y^2-4\,y=4\,x$

*16. $x=\dfrac{13+7\,y}{3}$ and $9\,x^2+49\,y^2-85=0$

27, C. Long Division

Divide. Check by multiplication or by numerical substitution.

1. $x^2 + 8x + 15$ by $x + 5$ 2. $x^2 - 30 - x$ by $x - 6$ (Do you think it advisable to rearrange the terms in the dividend of example 2 before dividing?

3. $a^4 - 9a^2 + a^3 - 16a - 4$ by $a + 2$

4. $a^3 - b^3$ by $a - b$ 5. $a^3 + 8b^3$ by $a - 2b$

6. $6x^3 - 17x^2 + 2x + 15$ by $2x - 3$

7. $x^3 + x^2 - 30x$ by $x + 5$ 8. $x^{2a} - y^{2b}$ by $x^a - y^b$

9. $a^3 + b^3$ by $a + b$ 10. $x^5 - y^5$ by $x + y$

11. $x^3 - 19x + 30$ by $x + 5$

Find by division whether

12. $3x + y - 2z$ is the square root of
$$9x^2 + 4z^2 + y^2 + 6xy - 12xz - 4yz$$

13. $x^m + 2x^n$ is the square root of $x^{2m} + 8x^{m+n} + 4x^{2n}$

*14. $2a - 3b^2$ is the cube root of $8a^3 - 36a^2b^2 + 54ab^4 - 27b^6$

27, D. Classification by Exponents

For each equation or expression below give the degree in x, in y, and in x and y.

* 1. $x^2 + 5x - 7 = 0$ *2. $x^4 - 7x^2 - 10 = 0$

* 3. $x^6 - 4x^2 - 6 = 0$ *4. $x^3 - y^3 = 26$

* 5. $xy = 8$ *6. $x^3 - 2x^2y^2 - y^2$ *7. $3x^2y - 4xy^2 + 7x^2y^2$

* 8. $2\pi xy$ *9. $(x + 3)(x - 1)$

Exercise 28. Negative, Fractional, and Zero Exponents

(See page 104.)

28, A. Zero Exponents and Negative Exponents

1. What is the meaning of x^0? Why has it been given this meaning?

2. What is the meaning of x^{-a}? Why has it been given this meaning?

Give the simplest value of:

3. $5^0 \qquad b^0 \qquad (a+b)^0 \qquad 4\,a^0b^0 \qquad x^5 \div x^5 \qquad 5\,x^0$

4. $\dfrac{x^3}{x^0} \qquad \dfrac{x^0}{x^2} \qquad z^3y^0x^2 \qquad (a^2b^2c^3)^0 \qquad (5^0)^2 \qquad 2^{-4}\cdot 3^{-3}\cdot c^0$

For each fraction find a multiplier which will remove the negative exponents: thus, $\dfrac{a^{-1}}{b^{-2}} \times \dfrac{ab^2}{ab^2} \equiv \dfrac{b^2}{a}$

5. $\dfrac{r^{-1}}{s^{-3}} \qquad \dfrac{1-r^{-1}}{s^{-2}} \qquad \dfrac{r^{-1}s^{-2}}{t^{-3}} \qquad \dfrac{r^{-1}+s^{-2}}{t^{-3}} \qquad \dfrac{r^{-2}}{s^{-3}-t^{-4}}$

6. $\dfrac{a^{-2}b^{-3}}{5^{-1}} \qquad \dfrac{5\,a^{-1}b^{-2}}{3} \qquad \dfrac{5^{-1}+a^{-1}}{c^{-1}} \qquad \dfrac{3(a^{-1}-b^{-2})}{5\,c^{-1}+e^{-2}}$

7. Write without denominators:

$\dfrac{a}{b} \qquad \dfrac{a^2}{b^2} \qquad \dfrac{1}{b^2c^3} \qquad \dfrac{3}{5\,ab^2c} \qquad \dfrac{a^{-1}}{3\,b} \qquad \dfrac{(a)^2}{(b^2)} \qquad \dfrac{a}{(2\,b)^{-1}}$

$\dfrac{x}{x^{-2}y^{-3}} \qquad \dfrac{a^{-3}}{b^{-2}} \qquad \dfrac{2}{a^{-1}-b^{-2}}$

8. Express the x's with negative exponents:

$\dfrac{a}{x^3} \qquad \dfrac{a}{bx^3} \qquad \dfrac{x^2y}{a^2b^3} \qquad \dfrac{3\,a}{2\,b^2x} \qquad \dfrac{3\,x^3b}{3\,x^2c} \qquad \dfrac{x^4}{x^3}$

Give the simplest value of:

9. 4^{-2} 3^{-2} 5^{-1} $(-6)^{-2}$ $(-4)^{-3}$ $(-\frac{1}{2})^{-2}$

10. 10^3 10^2 10^1 10^0 10^{-1} 10^{-2} 10^{-3}

Perform the operations indicated:

11. $(x^0+y)(x^0-y)$ $(2\,a^0b-3\,ab^0)^2$

12. $(x^{-1})^2$ $(x^{-1})(x^{-2})$ $(x^{-1}+y^{-1})(x^{-1}-y^{-1})$ $(x^{-2})^{-2}$

13. $(1-8\,b+15\,b^2)\div(a^0-3\,b)$

14. $(1+2\,y+y^2)\div(x^0+y)$

15. $5^0\,(x^{-2})^{-2}(-y^{-3})^{-3}$ $(a^0-b^{-2})(a^{-2}+b^0)$

***16.** $3^2-2\cdot3^2+5\cdot2^1-3\cdot4^0\,(-3)^{-2}+2\cdot3^{-1}\div3+2(3^2-2^3)$

Transform into equivalent fractions without negative exponents:

17. $\dfrac{a^{-2}+b^{-3}}{a^{-3}-b^{-2}}$ **18.** $\dfrac{a^{-3}b^{-4}}{2\,ab^2}$ **19.** $\dfrac{x^{-1}+y^{-2}}{x^{-1}y^{-2}}$

20. $\dfrac{2\,x^{-1}+5\,y^{-1}}{3\,x^{-1}-4\,y^{-1}}$ **21.** $\dfrac{a^{-2x}b^{-y}}{2\,a^{-x}b^{2y}}$ ***22.** $\dfrac{x^{-r}+y^{-r}}{x^{-r}-y^{-r}}$

28, B. Fractional Exponents (See page 105.)

1. We now have two methods of indicating that a root is to be taken. What are they?

2. Indicate (1) with the help of a radical sign, and (2) with the help of a fractional exponent, that b is to be cubed and that the square root of the result is to be taken.

Express with fractional exponents:

3. $\sqrt{2}$ $3\sqrt{2}$ $\frac{1}{2}\sqrt{3}$ $a\sqrt{b^3}$ $c\sqrt{d^2c^3}$ $\sqrt{8\,a^2b^5}$ $\sqrt{\frac{3}{4}}$

4. $\sqrt{\dfrac{x}{9}}$ $\frac{1}{3}\sqrt{3}$ $\dfrac{1}{3\,x}\sqrt{x^5}$ $\frac{1}{2}\sqrt{2\,a}$

$\sqrt[3]{8\,x}$ $\frac{2}{3}\sqrt{27\,a}$ $\sqrt{16\,a^2}$

*5. $a\sqrt{4\,a^4}$ $\quad b^2\sqrt{9\,b^5}$ $\quad \frac{3}{2}\sqrt{27\,a^2}$ $\quad \frac{1}{2}\sqrt{8\,a^5}$ $\quad \sqrt[4]{x^3}$ $\quad \sqrt[5]{(ab)^2}$

Express with radical signs:

6. $2\,a^{\frac{1}{2}}$ $\quad (3\,a)^{\frac{1}{2}}$ $\quad (4\,b^3)^{\frac{1}{2}}$ $\quad 3^{\frac{1}{2}}a^{\frac{1}{2}}$ $\quad ab^{\frac{1}{2}}$ $\quad 5^{\frac{1}{2}}a$ $\quad \left(\frac{8}{a}\right)^{\frac{1}{2}}$ $\quad a^{\frac{1}{3}}$

7. $2\,a^{\frac{1}{3}}$ $\quad (5\,a)^{\frac{1}{3}}$ $\quad a^{\frac{2}{3}}$ $\quad 2\,a^{\frac{2}{3}}$ $\quad 4^{\frac{1}{3}}\,b^{\frac{2}{3}}$ $\quad 6^{\frac{2}{3}}$ $\quad x^{\frac{3}{4}}$ $\quad a^{\frac{2}{3}}b^{\frac{1}{4}}$

Find the simplest value of (give principal roots only):

8. $(.25)^{\frac{1}{2}}$ $\quad (.09)^{\frac{3}{2}}$ $\quad (\frac{1}{4})^{\frac{3}{2}}$ $\quad 4^{-\frac{1}{2}}$ $\quad 4^{-2}$ $\quad (.0016)^{-\frac{3}{4}}$ $\quad (-8)^{-\frac{1}{3}}$

9. $9^{.5}$ $\quad 16^{.5}$ $\quad 16^{1.5}$ $\quad 25^{-\frac{3}{2}}$ $\quad 8^{\frac{2}{3}}$ $\quad (\frac{9}{4})^{-\frac{1}{2}}$

Express with one exponent:

10. $10^3 \cdot 10$ $\quad 10^{.5} \cdot 10^{1.5}$ $\quad 10^{3.2} \cdot 10$

$10^{4.261} \cdot 10$ $\quad 10^{5.261} \cdot 1$

Perform the operations indicated. In Examples 11–15 check your results by replacing the fractional exponents with radical signs and repeating the operations.

11. $(a^{\frac{1}{2}} - b^{\frac{1}{2}})(a^{\frac{1}{2}} + b^{\frac{1}{2}})$

***12.** $(a^{\frac{3}{2}} - b^{\frac{3}{2}})(a^{\frac{3}{2}} - b^{\frac{3}{2}})$

13. $(a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}})(a^{\frac{1}{3}} + b^{\frac{1}{3}})$

***14.** $(a^5 - b^5) \div (a^{\frac{5}{2}} - b^{\frac{5}{2}})$

15. $(a^{\frac{1}{2}} - b^{\frac{1}{2}})^3$

***16.** $(a^3 - b^3) \div (a^{\frac{3}{2}} + b^{\frac{3}{2}})$

***17.** $(a^{\frac{7}{2}} - 8\,b^{\frac{11}{2}}) \div (a^{\frac{7}{6}} - 2\,b^{\frac{11}{6}})$

***18.** $(a^{\frac{3}{2}} - 5\,ab^{\frac{1}{2}} + 9\,a^{\frac{1}{2}}b - 9\,b^{\frac{3}{2}}) \div (a^{\frac{1}{2}} - 3\,b^{\frac{1}{2}})$

***19.** $(a^{\frac{3m}{2}} - a^m b^{\frac{n}{2}} - a^{\frac{m}{2}}b^n + b^{\frac{3n}{2}}) \div (a^m - b^n)$

***20.** $(a^{2m} + a^m b^{2n} + b^{4n}) \div (a^m - a^{\frac{m}{2}}b^n + b^{2n})$

Exercise 29. Exponents, General

(See page 105.)

1. State in symbols the four laws of exponents. Give a numerical illustration of each law.

2. Give the meaning of the expressions that follow and justify each one:

$$a^0 \qquad a^{\frac{1}{2}} \qquad a^{\frac{1}{3}} \qquad a^{\frac{2}{3}} \qquad a^b \qquad a^{\frac{l}{r}} \qquad a^{-1} \qquad a^{-b} \qquad \frac{1}{a^{-2}}$$

Give the numerical value of each expression 3–11:

3. $2.84 \times 10^5 \qquad 3.76 \times 10^{-2} \qquad 1.06 \times 10^{-5}$

4. $2^5 \qquad 2^0 \qquad 2^{-1} \qquad 2^{-5} \qquad 2^{\frac{1}{2}} \qquad 2^{\frac{1}{3}} \qquad 2^{1.5}$

5. $\left(\frac{16}{25}\right)^{\frac{1}{2}} \qquad (2\tfrac{1}{4})^{\frac{1}{2}} \qquad \left(\frac{.04}{.09}\right)^{\frac{1}{2}} \qquad (.008)^{-\frac{1}{3}} \qquad (-.027)^{-\frac{2}{3}}$

6. $8^{\frac{2}{3}} - 64^{\frac{1}{6}} + 25^{-\frac{1}{2}} + \frac{1}{4^{-1}}$

7. $\frac{(.25)^{\frac{1}{2}} \times 0.008^{\frac{1}{3}}}{(0.16)^{-\frac{1}{2}}}$

8. $9^{\frac{3}{2}} + (-32)^{\frac{4}{5}} + 144^0 + 81^{-\frac{1}{2}}$

9. $\sqrt[3]{10^6} \qquad \sqrt[10]{10^5}$

10. $10^{-5} \qquad 10^{-4} \qquad 10^{-1} \qquad 10^0 \qquad 10^2 \qquad 10^5$

11. $1^{1000} \qquad 1^0 \qquad 1^{-2} \qquad 1^{\frac{1}{2}} \qquad 1^{1.5}$

12. Express without negative or zero exponents:

$$x^3 \qquad 2\,a^{-2} \qquad 7\,m^2n^{-2} \qquad \frac{4}{a^{-2}} \qquad \frac{4c^0}{ab^{-2}} \qquad \frac{a^{-2}b^3c}{ab^{-2}c^{-3}}$$

Perform the operations indicated and express the results in some simple form. If division is indicated in fractional form, write without denominator, or at least with no negative exponents in the denominator.

13. $a^{\frac{1}{2}} \cdot a \qquad a^{\frac{1}{2}} \cdot a^{\frac{3}{2}} \qquad b^{\frac{1}{3}} \cdot b^{\frac{1}{4}} \qquad s^2 \cdot s^{\frac{1}{2}} \cdot s^{\frac{1}{4}}$

$x^{\frac{3}{4}} \div x^{\frac{1}{4}} \qquad x^{1.4} \cdot x^{1.3} \cdot x^{2.1}$

14. $2^2 \cdot 2^3 = 2^? \qquad 2^2 \cdot 4 = 2^? \qquad 2^n \cdot 2^{2n} = 2^?$

15. $a^n \cdot a \qquad a^{n-1} \cdot a \qquad a^{n-2} \cdot a \qquad a^{n+1} \cdot a \qquad a^{n+2} \cdot a^2$

16. $a^n \div a \qquad a^{n+1} \div a^2 \qquad a^{n-1} \div a^{n-2} \qquad a^2 \div a^n \qquad a^n \div a^{n-2}$

17. $\sqrt{a^2b^{-4}c^6} \qquad \sqrt{a^{2n-2}} \qquad \sqrt{a^2b^{\frac{1}{3}}c^{-4}} \qquad \sqrt[3]{\frac{8\,ab^{-\frac{1}{2}}}{c}}$

18. $b^5 \cdot b^{-3}$ $(a^{\frac{2}{3}}b^{\frac{1}{2}})(7\,ab^{-\frac{5}{2}})$ $a^3 \div a^{-2}$ $a^{-3} \div a^2$

19. $(a^{-1}b^{\frac{1}{2}}c^{-\frac{3}{4}})^4$ $(x^{-2}y^{\frac{1}{3}}z^{-3})^{-\frac{3}{2}}$

20. $\dfrac{27^{\frac{2}{3}}\,ab^3}{2^{-1}a^{-1}b^{-3}}$ $\dfrac{5^{-2}\,(m-n)^0b^{-2}}{b^{-3}y^0}$ $\dfrac{16^{-\frac{5}{4}} \cdot 27^{-\frac{4}{3}}}{64^{-\frac{4}{3}}\,9^{-\frac{1}{2}}}$

21. $(4^{-3} \cdot 3^{-4} \cdot 2^{-3}) \div (16^{-\frac{5}{4}} \cdot 81^{-\frac{1}{2}} \cdot 27^{-\frac{2}{3}})$

22. $a^2b \cdot a^{-1}b^2 \cdot ab^{-1}$ $x^{3a-2} \cdot x^{a-3}$

*23. $(3^{n+2} + 3 \cdot 3^n) \div (3^{n+1} \cdot 9)$

*24. $\left(\dfrac{2}{5}\right)^{-3} + \dfrac{5}{2^{-2}} + (-2)^{-2} + 16^{-\frac{3}{4}}$

*25. $(2 \cdot 4^{-5})^{\frac{1}{3}} - \dfrac{2^{\frac{1}{7}}}{2^{-\frac{6}{7}}} - 128^{-\frac{3}{7}} + \dfrac{1}{8^{-\frac{4}{3}}}$

*26. $(ab^2c^{2x})^x$ $(a^{-\frac{3}{4}}b^{-9}c^6)^{-\frac{2}{3}}$ $\sqrt{x^{10}y^{-4}z^{-2}}$

27. $\sqrt[3]{-8\,a^{12}b^{-9}c^{-3}}$ $\sqrt{\dfrac{x^{-2}y^4c^{-8}}{a^4b^{-6}}}$ $\dfrac{(a^{\frac{1}{2}}bc^0)^4}{(a^{-1}bc^2)}$

*28. $\dfrac{(x^6y^9z^{-3})^{\frac{2}{3}}}{(8\,a^{12}b^{-6})}$ $\left(\dfrac{x^8y^{-12}z^{-4}}{16\,b^{\frac{3}{4}}c^0}\right)^{-\frac{3}{4}}$

29. $\dfrac{3}{a^{-2} - b^{-2}}$ $\dfrac{4}{b^{-3} - c^{-3}}$ $\dfrac{a^{-1}}{b^{-2}c^{-3}}$ $\dfrac{a^{-1}}{b^{-2} + c^{-3}}$

*30. $\dfrac{2}{2^{-1} + 2^{-2}}$ $\dfrac{2^{-1} + 3^{-1}}{2^{-1} - 3^{-1}}$ $\dfrac{a^{-1} + 2\,b^{-1}}{a^{-3} + 8\,b^{-3}}$

31. $a^l \cdot a^s$ $a^l \div a^s$ $(a^l)^p$ $\sqrt[r]{a^l}$

Exercise 30. Numbers Represented by Powers of 10

(See page 108.)

Object: To teach a practical use of exponents and to prepare for the use of logarithms.

Give the value of:

1. 3.70×10^0 5.29×10 6.82×10^2

2. 4.26×10^{-1} 6.23×10^{-3} 7.954×10^{-5}

3. 8.888×10^{-6} 9.999×10^{6} 7.77×10^{0}

4. Represent by powers of 10 each number of Exercise 32, A.

Try the tests on pages 108–111.

Exercise 31. Finding Characteristics of Logarithms

(See page 117.)

1. Give the characteristic of the logarithm of each number in Exercise 32, A.

2. Give the characteristic of the logarithm of each number in Exercise 32, B.

Exercise 32. Finding Logarithms of Numbers

32, A. No Interpolation

(See page 117.)

1. Why is it important to think of a logarithm as a binomial?

Find in the table the logarithms of the following numbers:

2. 357 **3.** 35.7 **4.** 826 **5.** 82.6 **6.** 8.26

7. 0.026 **8.** 87,200 **9.** 1.25 **10.** 2.29 **11.** 0.134

12. 0.345 **13.** 0.567 **14.** 0.892 **15.** 0.999 **16.** 0.0327

17. 0.0528 **18.** 0.0693 **19.** 0.00777 **20.** 0.000444

21. 5 **22.** 50 **23.** 505 **24.** 7.07 **25.** 990 **26.** 1

27. 2 **28.** 0.0101 **29.** 2.32×10^{4} **30.** 9.45×10^{-3}

32, B. With Interpolation. (See page 118.)

1. Show that changes in logarithms in the table are nearly but not quite proportional to changes in corresponding numbers. Can you find log 25 by interpolating between log 20 and log 30? Explain.

2. If a number consists of five or more digits, how do you proceed to find its logarithm in the table?

Find the logarithms of the following numbers. Give each mantissa to the nearest fourth figure.

3. 2712 **4.** 3431 **5.** 4338 **6.** 43.28 **7.** 0.4328 **8.** 5.555

9. 0.6666 **10.** 576.4 **11.** 4.183 **12.** 6426 **13.** 7,185,000

14. 7.878 **15.** 31.08 **16.** 240.9 **17.** 50.37 **18.** 0.8943

19. 0.9342 **20.** 0.3429 **21.** 4.002 **22.** 70.09 **23.** 86.471

24. 324.37 **25.** 45.213 **26.** 530.08 **27.** 69,402 **28.** 0.06241

29. 0.007432 **30.** 0.00062 **31.** 0.08888 **32.** 0.09999

33. 9.36×10^9 **34.** 4.826×10^{-6} **35.** 8.7549×10^{12}

Exercise 33. Finding Antilogarithms. No interpolation

(See page 119.)

33, A

Give the correct position of the decimal point in each antilogarithm below. Check each result by applying the law for characteristics. ($\bar{1}.9547$ is often written $9.9547-10$.)

1. log (547) = 1.7380 **2.** log (285) = 2.4548

3. log (850) = 3.9294 **4.** log (805) = 2.9058

5. log (115) = 0.0607 **6.** log (470) = 0.6721

7. log (956) = 4.9805 **8.** log (105) = 5.0212

9. log (901) = $\bar{1}.9547$ **10.** log (779) = $\bar{3}.1589$

11. log (101) = $\bar{2}.0043$ **12.** log (204) = $\bar{5}.3096$

13. log (790) = 1.8982 **14.** log (585) = $\bar{1}.7672$

15. log (600) = 2.7782 **16.** log (100) = $\bar{3}.0000$

17. log (102) = 2.0086 **18.** log (999) = 0.9996

19. log (201) = 1.3032 **20.** log (210) = $\bar{4}.3222$

33, B

Copy the following logarithms and find their antilogarithms. Roughly check the work by checking the size of each number by reference to exact powers of 10:

1. log (   ) = 1.4014
2. log (   ) = 3.5502
3. log (   ) = 0.6599
4. log (   ) = $\bar{1}$.7067
5. log (   ) = $\bar{2}$.7810
6. log (   ) = 4.7404
7. log (   ) = 1.9435
8. log (   ) = $\bar{4}$.9542
9. log (   ) = $\bar{2}$.6117
10. log (   ) = 0.0128

Check each result, 1–10, by finding its logarithm.

Exercise 34. Finding Logarithms and Antilogarithms, with Interpolation

(See page 120.)

34, A

Find to the nearest third figure the antilogarithm of each number in 34, B.

34, B

Find to the nearest fourth figure the antilogarithm of each number below.

1. log (   ) = 1.7020
2. log (   ) = 2.6289
3. log (   ) = 0.4437
4. log (   ) = $\bar{1}$.6306
5. log (   ) = $\bar{2}$.8776
6. log (   ) = 2.9585
7. log (   ) = 0.7804
8. log (   ) = 1.6617
9. log (   ) = $\bar{3}$.8753
10. log (   ) = 0.8067
11. log (   ) = 1.5547
12. log (   ) = $\bar{1}$.5042
13. log (   ) = $\bar{3}$.6626
14. log (   ) = 0.8800

15. $\log (\quad) = 1.0062$ **16.** $\log (\quad) = \bar{2}.0623$

17. $\log (\quad) = 4.9730$ **18.** $\log (\quad) = \bar{1}.0409$

19. $\log (\quad) = 0.6901$ **20.** $\log (\quad) = 2.7403$

34, C

Object: To contrast the processes of finding logarithms and antilogarithms.

Supply the missing numbers.

1. $\log 4784 =$ **2.** $\log 32.99 =$

3. $\log (\quad) = 0.9317$ **4.** $\log (\quad) = \bar{2}.8192$

5. $\log 0.7777 =$ **6.** $\log (\quad) = 0.7777$

7. $\log 88.823 =$ **8.** $\log 4.003 =$

9. $\log (\quad) = \bar{2}.6024$ **10.** $\log (\quad) = \bar{1}.0210$

Exercise 35. Logarithmic Computation, Ordinary Cases

(See page 122.)

Why is it important to keep in mind the fact that a logarithm is an exponent?

In each computation below, give a preliminary estimate of the result; also tell in advance the number of significant figures which should be retained in the result; arrange your work systematically.

1. 89.6×33.3 **2.** 473×9.34 **3.** 216.7×-4.324

4. $\dfrac{576.4}{423.1}$ **5.** $\dfrac{832 \times -47.82}{-14.24}$ **6.** $(106)^3$

7. $\dfrac{0.0360 \times (55.7) \times 437}{45.67}$ **8.** $\dfrac{7234 \times 7472}{7132 \times 6178}$ **9.** $\sqrt{9431}$

10. $\sqrt[3]{75.69}$ **11.** $\dfrac{\sqrt{6381}}{3.238}$ **12.** $\left(\dfrac{4.82}{1.43}\right)^2$

13. $450 \times (1.05)^4$

14. $\sqrt[3]{\dfrac{467.3}{151}}$

15. $\sqrt[5]{\dfrac{30.09 \times 410.6}{18.94}}$

16. $\dfrac{0.451 \times 481.9 \times 0.600}{0.372 \times 0.935 \times 4.713}$

17. $\dfrac{7563 \times \sqrt[3]{-80.27}}{46.7 \times 437.4}$

18. $\dfrac{(\sqrt{287.2} \times -2.758)^2}{4.731}$

19. $\dfrac{3.265\sqrt{90.77}}{(4.266)^2}$

Exercise 36. Addition and Subtraction of Logarithms with Negative Characteristics

(See page 122.)

36, A

Add the logarithms below. Check by estimating the products and giving the characteristics of the logarithms of your estimates.

36, B

Subtract the logarithms below. Check by addition. It is also possible to use a check similar to that suggested in A.

1. $\log 22.2 = 1.3464$
   $\log 133 = 2.1239$

2. $\log 7.47 = 0.8733$
   $\log 39.1 = 1.5922$

3. $\log 1.39 = 0.1430$
   $\log 757 = 2.8791$

4. $\log 0.00287 = \bar{3}.4579$
   $\log 16.4 = 1.2148$

5. $\log 0.000150 = \bar{4}.1761$
   $\log 2.31 = 0.3636$

6. $\log 0.000376 = \bar{4}.5752$
   $\log 0.667 = \bar{1}.8241$

7. $\log 0.488 = \bar{1}.6884$
   $\log 6.21 = 0.7931$

8. $\log 7.65 = 0.8837$
   $\log 0.0856 = \bar{2}.9325$

9. $\log 0.0149 = \bar{2}.1732$
   $\log 0.0159 = \bar{2}.2014$

10. $\log 0.0119 = \bar{2}.0755$
    $\log 0.157 = \bar{1}.1959$

Exercise 37. Multiplication and Division of Logarithms with Negative Characteristics

(See page 123.)

37, A

Multiply the following logarithms as indicated. Check by division. ($\bar{1}.0267 \equiv 9.0267 - 10.$)

1. $\begin{array}{r}\bar{1}.0267\\ 4\\ \hline\end{array}$ 2. $\begin{array}{r}\bar{1}.2387\\ 6\\ \hline\end{array}$ 3. $\begin{array}{r}\bar{2}.5367\\ 5\\ \hline\end{array}$ 4. $\begin{array}{r}\bar{3}.3333\\ 4\\ \hline\end{array}$

5. $\begin{array}{r}\bar{4}.8176\\ 5\\ \hline\end{array}$ 6. $\begin{array}{r}\bar{3}.2614\\ 25\\ \hline\end{array}$ (Rewrite the logarithm as a binomial.)

37, B

Divide as indicated. When necessary rewrite the logarithm as a binomial in convenient form. Check by multiplication.

1. $6\overline{)0.3427}$ 2. $3\overline{)\bar{3}.2614}$ 3. $2\overline{)\bar{1}.3876}$

4. $8\overline{)\bar{2}.7348}$ 5. $12\overline{)\bar{5}.3428}$ 6. $7\overline{)\bar{4}.9932}$

Exercise 38. Logarithmic Computation. Special Cases

(See page 124.)

Estimate each result. Determine the number of significant figures in each result. Write the formula. Compute systematically. Numbers given to 1 or 2 figures only may be considered exact; all others approximate.

1. $\dfrac{16.51}{243}$ 2. $\dfrac{2.536}{5.738}$ 3. $\sqrt{0.05927}$

4. $\dfrac{17.82 \times 0.0843}{97.34}$ 5. $\sqrt{\dfrac{5.83 \times 38.72}{4.732}}$

6. $\sqrt[3]{\dfrac{47.93}{-1.467 \times 0.2932}}$

7. $\dfrac{4.973^2 \times \sqrt{6924}}{14.82}$

8. $\sqrt{48 \times 32\sqrt{6 \times 924}}$

9. $\dfrac{0.0760 \times 54.76}{(2.34)^2}$

10. $(1.06)^{25}$

11. $\dfrac{\sqrt[3]{40.12 \times 0.00981}}{\sqrt{3.821}}$

12. $\dfrac{\sqrt{0.3579} \times 0.2222}{66.06}$

13. $\dfrac{\sqrt{0.3579 \times 0.2222}}{66.06}$

14. $\sqrt{\dfrac{0.3579 \times 0.2222}{66.06}}$

***15.** $\sqrt[5]{\dfrac{\sqrt[3]{476.5} \times (3.81)^2}{(0.00230)^{\frac{1}{2}}}}$

***16.** $\dfrac{1.52^5 \times 0.00213\sqrt[3]{14.23}}{0.0516\sqrt{0.002531}}$

***17.** $\dfrac{3.765 \times 27.38}{(23.2)^5 \times (0.260)^2}$

***18.** $\dfrac{-2.876}{(0.0160)^2\sqrt{29.8}}$

***19.** $\sqrt[3]{\dfrac{32.94 \times 0.08637}{(0.0021536)\sqrt{30.00}}}$

20. $2500(1.04)^{12}$ ***21.** $S = \dfrac{ar^n - a}{r - 1}$ Find S if $a = 12$, $r = 2.5$ and $n = 8$.

22. $S = \frac{1}{2}\,g(2\,t - 1)$ Find S if $g = 32.16$ and $t = 44.30$.

23. $C = \dfrac{nE}{R + rn}$ Find n if $E = 1.578$, $C = 8.771$, $R = 1.825$, and $r = 0.0498$.

24. Find by logarithms the value of

$$\frac{1}{9} \qquad \frac{1}{3254} \qquad \frac{1}{76.34}$$

*25. The distance S that a body falls from rest in t seconds is given by the formula $S = 16\,t^2$. How long will it take a ball to fall from the top of a monument 555.5 ft. high?

Exercise 39. The Use of Logarithmic Tables of Trigonometric Functions (See page 133.)

Supply the numbers missing below:

1. $\log \sin 17.3° =$
2. $\log \cos 21.4° =$
3. $\log \sin 15°\,34'$
4. $\log \cos 49°\,53'$
5. $\log \cos 67.42° =$
6. $\log \tan 57.32° =$
7. $\log \sin 82.14° =$
8. $\log \cos 29.05° =$
9. $\log \sin (\quad) = \bar{1}.5602$
10. $\log \sin (\quad) = \bar{1}.7048$
11. $\log \cos (\quad) = \bar{1}.6942$
12. $\log \cos (\quad) = \bar{1}.8814$
13. $\log \tan (\quad) = \bar{1}.7670$
14. $\log \tan (\quad) = 0.7425$
15. $\log \sin (\quad) = \bar{2}.8920$
16. $\log \cos (\quad) = \bar{1}.0840$

Exercise 40. Solving Right Triangles by Logarithms (See page 136.)

Estimate the lengths of the missing sides; solve the triangles; check your solutions.

1. $b = 386, \quad A = 42.0°$
2. $a = 41.3', \quad B = 57°\,10'$
3. $c = 17.3', \quad B = 43.5°$
*4. $c = 42.9', \quad A = 78.4°$
5. $b = 21.6'', \quad B = 63°\,40'$
6. $a = 342', \quad A = 48.6°$
7. $a = 417', \quad c = 572'$
8. $a = 41.32', \quad b = 38.42'$
9. $b = 49.37', \quad a = 82.43'$
10. $c = 298.4', \quad a = 171.9'$
11. $a = .3259'', \quad B = 47.38°$
12. $b = .08462, \quad B = 15°\,20'$
*13. $a = .5555, \quad b = .7893$
*14. $b = 14320', \quad A = 14°\,43'$
*15. $c = .2371, \quad B = 68°\,14'$

EXERCISES FOR CHAPTER IV

PROGRESSIONS

Exercise 41. Arithmetic Progression

(See page 144.)

41, A

1. Define *sequence, series, arithmetic series.*

2. Tell how to determine whether a sequence is arithmetic.

3. State and derive three formulas for arithmetic progression.

4. What are the steps in a mathematical study, as stated on page 141 and illustrated on page 142?

41, B. Finding the nth Term

(See page 144.)

In each of the following find the term indicated:

1. $2, 5, 8 \ldots$	the 6th
2. $4, 9, 14 \ldots$	the 10th
3. $2, 17, 32 \ldots$	the 20th
4. $-5, -8, -11 \ldots$	the 18th
5. $-5, -2, 1 \ldots$	the 8th
6. $6, 3, 0 \ldots$	the 12th
7. $2, 2\frac{1}{3}, 2\frac{2}{3} \ldots$	the 19th
8. $.0025, .01, .0175 \ldots$	the 11th
9. $a - b, a - 2b, a - 3b$	the 5th
10. $n - 2, n, n + 2 \ldots$	the 7th
11. $2a + b, 3a, 4a - b \ldots$	the 7th

*12. The 11th term of the preceding sequence.

*13. $1\frac{1}{2}, 3, 4\frac{1}{2} \ldots$ the 100th

*14. $-5, -1, 3 \ldots$ the 30th

*15. $2\frac{1}{3}, 4, 5\frac{2}{3} \ldots$ the 10th

16. Which term of $2\frac{1}{3}, 3, 3\frac{2}{3}$, is 65?

17. Which term of .025, .1, .175 . . . is 1?

*18. Which term of $a - 2b, -b, -a \ldots$ is $8b - 9a$?

*19. Which term of $6, 3, 0 \ldots$ is -21?

*20. Which term of $x + \frac{1}{2}\pi, x - \frac{1}{2}\pi, x - \frac{3}{2}\pi \ldots$ is $x - \frac{11}{2}\pi$?

41, C. Finding the Sum

(See page 144.)

Find the sum of:

1. 10 terms of $1 + 5 + 9 + \ldots$
2. 12 terms of $3 + 4\frac{1}{2} + 6 + \ldots$
3. 16 terms of $0 - 2 - 4 - \ldots$
4. 30 terms of $.15 + .2 + .25 + \ldots$
5. 13 terms of $4 - 8 - 20 - \ldots$
6. 11 terms of $1 - 1 - 3 - \ldots$
7. 8 terms of $1 + \frac{1}{2} + 0 - \ldots$
8. 7 terms of $(a - 2) + (a - 1) + a + \ldots$
9. 12 terms of $6 + 3 + 0 - \ldots$

*10. 12 terms of $(a - 2) + a + (a + 2) + \ldots$

11. 9 terms of $2\frac{1}{3} + 3 + 3\frac{2}{3} + \ldots$

Of how many terms of:

12. $-5 - 2 + 1 + \ldots$ is 280 the sum?

Of how many terms of:

***13.** $(a - b) - b - (a + b) \ldots$ is $-9a - 6b$ the sum?

***14.** $-11 - 7.5 - 4. - \ldots$ is 202.5 the sum?

***15.** $3 + 3\frac{3}{4} + 4\frac{1}{2} + \ldots$ is 45 the sum?

41, D. Applications of Arithmetic Progression

(See page 144.)

For each problem below, write an A.P. and find the required numbers by use of formula:

1. After graduating from high school, a boy received a salary of $960 for the first year. If his annual increase is $50, how much will he receive for his tenth year? For the ten years?

2. A clock which strikes the hours only, strikes how many times in twelve hours? If a European railway clock strikes the hours from 1 to 24, how many times will it strike in 24 hours?

3. A man agreed to dig a well for 50 cents for the first foot, 60 cents for the second foot, 70 cents for the third foot, and so on. How much should he receive if the well is 20 ft. deep? 60 ft. deep?

4. A baseball thrown from a window near the top of Washington Monument falls 16 ft. the first second, and 32 ft. farther in each succeeding second. How far does it fall in 5 seconds?

5. A car going down an incline goes 4 ft. the first second, 6 ft. the second second, 8 ft. the third second, and so on. How far does it go in a minute?

6. A king said to his blacksmith, "I will pay you for shoeing my horse 1 cent for the first nail, 3 cents for the second, 5 cents for the third, and so on; or 25 cents for each nail." If there were 8 nails in each shoe, which plan would pay the blacksmith better? How much?

*7. A body falls 16 ft. in the first second, 3 times as far the second second, 5 times as far the third second, and so on. How far does it fall in one minute?

8. What term of the sequence $-8, -5, -2 \ldots$ is 49?

*9. What term of the sequence $1.8, 2.4, 3 \ldots$ is 11.4?

*10. What term of the sequence $-1, -8, -15 \ldots$ is -78?

41, E. Inserting Arithmetic Means

(See page 146.)

1. What is the arithmetic mean between 8 and 17?

2. What is the arithmetic mean between -8 and 10?

3. Find the arithmetic mean between a and b.

Insert between each of the following terms the number of means specified:

4. $6 \ldots 14$	3 means
5. $\frac{46}{3} \ldots -\frac{8}{3}$	5 means
6. $\frac{1}{2} \ldots -\frac{13}{10}$	8 means
7. $3.6 \ldots 1.2$	5 means
8. $800 \ldots 70$	9 means
9. $(a-b) \ldots (16a+9b)$	4 means
10. $(x-y) \ldots (x+y)$	1 mean
*11. $a \ldots l$	4 means
12. $7\frac{1}{2} \ldots 30$	9 means
13. $1.7 \ldots 3$	12 means
*14. $(1+x) \ldots (1+13x)$	5 means
*15. $\frac{a}{2} \ldots \frac{b}{2}$	4 means

41, F. Transforming the Formulas and Using Them Indirectly

(See page 146.)

Given Formulas I, II, and III:

1. Derive a formula for a in terms of n, l, and s.

*2. Solve Formulas I and II as a pair of equations in a and l.

*3. Derive a formula for n in terms of a, d, and l.

4. Derive a formula for d in terms of a, l, and s.

Find the missing numbers:

	l	a	n	d	s
5.	26	-13	14	..	..
6.	..	..	35	$-\frac{3}{4}$	$-306\frac{1}{4}$
7.	-3	..	..	$-\frac{2}{3}$	13
* 8.	$a+b$	$a-b$	5	..	..
9.	23	1	..	..	120
10.	8	..	8	..	44
*11.	..	$a-\sqrt{2}$	..	$a+\sqrt{2}$	$14(2a+\sqrt{2})$
*12.	..	$\frac{a+1}{a}$	a	$\frac{1}{a}$	

13. Solve Formula I for d and use the resulting formula in finding 5 arithmetic means between 3 and 15.

*14. Use the same formula in finding 5 means between a and $6b-5a$.

41, G. Problems. Arithmetic Progression

(See page 148.)

1. The difference of two numbers is 3, and their arithmetic mean is 5. Find the numbers.

2. How many arithmetic means are inserted between $-\frac{3}{2}$ and $\frac{9}{2}$ if the sum of these means is $\frac{21}{2}$?

3. The sum of the 1st and 7th terms of an A.P. is 0; the sum of the 5th and 6th terms is 11. Find the first three terms.

4. The sum of the 1st and 4th terms of an A.P. is 19, and of the 3d and 6th, 31. What is the 1st term?

5. An A.P. consists of 21 terms. The sum of the three middle terms is 129 and of the last three 237. Find the series.

6. The sum of the first 7 terms of an A.P. is 105, and the sum of the 3d and 5th is 10 times the 1st term. Find the series.

***7.** The first term in each of two A.P.'s is 8; the difference in the second series is twice that in the first; and the sum of the first five terms of the second is $\frac{4}{3}$ that of the first. Find both series.

8. An automobile standing on a hill starts rolling down and rolls 3 ft. the first second and 3 ft. farther in each succeeding second. How far will it roll in 8 seconds?

***9.** A heavy body falling from rest will fall approximately $16\frac{1}{12}$ ft. the first second, $48\frac{1}{4}$ ft. the second second, $80\frac{5}{12}$ ft. the third second, and so on. Find the height to the nearest foot of a building if an object dropped from the top reaches the ground in 6 seconds.

10. In a certain contest \$2750 are awarded in ten cash prizes, each prize differing from the next by \$50. Find the amount of the smallest and of the largest prize.

11. How long will it take to pay off a debt of \$450 if I pay \$10 the first month and increase this by \$10 each succeeding month until the debt is canceled?

***12.** Deposits of \$50 at the beginning of each year with simple interest at 7% will amount to how much at the end of 15 years?

***13.** An \$8000 mortgage is to be paid off by annual payments of \$250 with 5% interest. What is the total amount to be paid to cancel the mortgage?

***14.** A \$4000 mortgage is to be paid off by monthly payments of \$40 with 6% interest. What is the total amount to be paid to cancel the mortgage?

***15.** If an automobile could increase its speed 1 ft. per second each second, how far could it go in one minute starting from rest? In 2 minutes? In how many seconds could it reach a speed of 60 miles an hour?

***16.** An agent arranges to sell a radio for \$360 to be paid in 20 monthly installments which form an arithmetic progression. It is also desired that the radio shall be two thirds paid for with the payment of the tenth installment. What will be the first and second payments?

***17.** Show that the sum of the first n consecutive odd numbers is equal to the square of an integer.

***18.** Show that the sum of the first n integers plus the sum of the first $n + 1$ integers is a square number.

***19.** A \$3600 debt was to be paid off in 40 monthly payments which formed an arithmetic progression. After 30 payments had been made, two thirds of the debt was paid, but the debtor found himself unable to continue the regular increases. Thereupon he offered to pay \$9 per month less than the last monthly payment plus 5% of the new payment each month as interest. How long did it take finally to clear the debt, and how much money did the creditor receive as interest?

Exercise 42. Geometric Progression

(See page 153.)

42, A

1. Define *geometric sequence.*

2. Tell how to determine whether a sequence is geometric.

3. Indicate the first five G.P.'s on page 141.

4. State and derive three formulas for geometric progression.

***5.** In finding the sum of a G.P., what are the advantages of indicating the result as the sum (or difference) of two fractions?

42, B. The *n*th Term. Without Logarithms

Find the terms indicated. The table on page 112 may occasionally be of use.

1. $1, 3, 9 \ldots$	the 6th	**2.** $2, 6, 18 \ldots$	the 8th
3. $45, -15, 5 \ldots$	the 5th	**4.** $1, \frac{1}{2}, \frac{1}{4} \ldots$	the 7th
5. $\frac{2}{3}, 2, 6 \ldots$	the 6th	**6.** $-s, s^2, -s^3$	the 9th
7. $1, -2x, 4x^2$	the 9th	***8.** $a^2, ab, b^2 \ldots$	the 10th
9. $1, -\frac{1}{a}, \frac{1}{a^2} \ldots$	the 9th	***10.** $x^{\frac{1}{2}}, x, x^{\frac{3}{2}} \ldots$	the 6th
***11.** $a^{\frac{1}{4}}b^{\frac{1}{4}}, a^{\frac{3}{4}}b^{\frac{3}{4}}, a^{\frac{5}{4}}b^{\frac{5}{4}} \ldots$	the 7th	***12.** $3, 3\sqrt{3}, 9 \ldots$	the 7th

42, C. The *n*th Term. With Logarithms

Assume that the given numbers are exact and work to the degree of accuracy permitted by the tables.

1. $3 + 9 + 27 + \ldots$	the 15th
2. $1 - \frac{1}{3} + \frac{1}{9} - \ldots$	the 13th
3. $1.06 + 1.06^2 + 1.06^3 + \ldots$	the 4th

4. $300(1.04) + 300(1.04)^2 + 300(1.04)^3 + \ldots$ the 6th

***5.** $650 + 650(1.03) + 650(1.03)^2 + \ldots$ the 12th

42, D. The Sum. Without Logarithms

Find by formula the sums:

1. $1 + 3 + 9 + \ldots$ 6 terms

2. $2 - 6 + 18 - \ldots$ 5 terms

3. $1 + \frac{1}{2} + \frac{1}{4} + \ldots$ 8 terms

***4.** $\frac{2}{3} + 2 + 6 + \ldots$ 8 terms

5. $-s + s^2 - s^3 + \ldots$ 8 terms

6. $1 - 2x + 4x^2 - \ldots$ 8 terms

***7.** $1 - \frac{1}{a} + \frac{1}{a^2} - \ldots$ 8 terms

***8.** $1 - 4 + 16 - \ldots$ 7 terms

***9.** $27 + 18 + 12 + \ldots$ 8 terms

10. $\frac{1}{4} + \frac{1}{2} + 1 + \ldots$ 9 terms

***11.** $-81 + 27 - 9$ 4 terms

12. $3 + 3\sqrt{3} + 9 + \ldots$ 8 terms Give result in simplest radical form.

***13.** $\frac{b}{a} + 1 + \frac{a}{b} + \ldots$ 7 terms

***14.** $0.1732 - 0.05196 + 0.015588$ 4 terms

42, E. The Sum. With Logarithms

(See directions for Exercise 42, C.)

1. $5 - 20 + 80 - \ldots$ 14 terms

2. $2 + 1 + \frac{1}{2} + \ldots$ 11 terms

3. $.3 + .03 + .003 + \ldots$	11 terms
4. $0.1732 - 0.05196 + 0.015588 - \ldots$	9 terms
5. $.62 + .0062 + .000062 + \ldots$	9 terms
***6.** $6 + 36 + 216 + \ldots$	8 terms
***7.** $3 - 9 + 27 - \ldots$	13 terms
8. $1.06 + 1.06^2 + 1.06^3 + \ldots$	10 terms
***9.** $650 + 650(1.03) + 650(1.03)^2 + \ldots$	15 terms
***10.** $1 - 8 + 64 - \ldots$	15 terms

42, F. Inserting Geometric Means. (See page 154.)

1. Define a *geometric mean.* Give another name for a single geometric mean.

2. Find three geometric means between 1 and 81 and show that two sets of answers are possible.

Find a geometric mean:

3. Between 1 and 64.

4. Between $12\frac{1}{4}$ and 1.

5. Between 5.76 and 0.09.

6. Between $\frac{1}{4}m$ and $9m^3$.

***7.** Between -8 and -32.

8. Between $\dfrac{5x^2y^5}{a}$ and $\dfrac{45a^{15}}{x^4y}$

***9.** Find one geometric mean between the approximate numbers 1.26 and 7.35. Give the answer to the nearest third figure.

Insert:

10. Four geometric means between $40\frac{1}{2}$ and $5\frac{1}{3}$.

11. Eight geometric means between $-\frac{5}{128}$ and 20.

12. Three geometric means between 1 and 0.0256.

*13. Four geometric means between 3072 and 3.

*14. Five geometric means between 1944 and $\frac{1}{24}$.

*15. One geometric mean between $3x^2$ and $\frac{1}{4}x$.

42, G. Indirect Use of the Formulas

1. Derive a formula for l in terms of a, r, and s.

2. Derive a formula for a in terms of r, l, and s.

3. Derive a formula for s in terms of r, n, and l.

Supply the missing numbers:

	a	r	n	l	s
4.	8	2	..	..	248
5.	..	6	5	1296	..
*6.	$\frac{8}{9}$	..	..	$-\frac{243}{16}$	$-8\frac{109}{144}$
*7.	200	1.03	9	..	..
*8.	450	..	11	665.3	..

42, H. Problems. Geometric Progression.

(See page 156.)

1. The last term of a geometric progression of 8 terms is -384 and the third term is 12. Find the first term and the sum of the series.

2. Find four numbers in geometric progression such that the sum of the first two is 16 and the sum of the last two is 144.

3. The sum of three numbers in geometric progression is $21\frac{2}{3}$ and their product is 125. Find the numbers.

4. The sum of the first and third terms of a geometric progression is 50 and the sum of the second and fourth is -150. Find a and r.

***5.** In starting a chain letter, Mrs. Hamilton contributed $1 to a charity and wrote three letters to three friends asking each of them to contribute the same amount and to send three similar letters. If ten sets of letters, including Mrs. Hamilton's, are written, and if all respond, how much money will the charity receive? Comment on the danger of the chain-letter plan.

***6.** A person who receives a copy of a certain chain letter is to write 5 similar letters. How many letters will the tenth set contain? How many letters will have been written in the ten sets? (Count the first letter as the first set.)

***7.** Each stroke of an air pump removes one tenth of the air in the receiver. What part of the air originally in the receiver remains after five strokes? What part has been removed?

8. If there were 2 people in the first generation, 4 in the second, 8 in the third, and so on, how many were there in the tenth? How many in the generation which came at the end of 3000 years if the generations are 30 years apart? How many people in all the generations of the 3000 years?

***9.** The population of continental United States in 1920 was 105,700,000. If the population increases 50% every forty years, what will it be in the year 2000?

***10.** The city of Albany, New York, had a population of 100,000 in 1910 and 113,000 in 1920. If the rate (per cent) of increase per decade is constant, what will be the population in 1960?

***11.** At 5% interest, $500 will amount to how much in 15 years if the interest is compounded annually? semi-annually?

***12.** When Jane Aubin was born, her grandmother placed $1000 in a savings bank to her credit. If the money remained in the bank at 4% compounded quarterly, what did it amount to when Jane was 17? 25? 40?

*13. At 4% interest compounded semi-annually, $700 put in the bank on the day a boy is born will amount to how much for college expenses at the end of 19 years?

*14. A man wishing to establish a large fund put $5000 in a savings bank where interest at 4% was compounded annually, and directed that it remain there 150 years. What was the amount at the end of the 150th year?

*15. A man bought a house for $15,000, paying 20% down and agreeing to pay the balance at the rate of $1000 every six months with interest of 8% on money due. How much did the house finally cost him?

*16. A boy about to begin a seven-year course in college and medical school has $4000. He hopes by earning an increasing amount of money to use each year only 15% of his capital. If he is able to do this, and so uses each year only 15% of what was left at the beginning of the year, how much of the $4000 (not counting interest) will he have left at the end of his course?

Exercise 43. Infinite Geometric Series

(See page 159.)

43, A

1. What is an infinite geometric series?

2. On what condition is an infinite geometric series said to have a sum?

*3. What is the criterion of a limit as the word *limit* is used in this chapter?

4. Derive a formula for the sum of an infinite geometric progression.

43, B. Finding the Sum

Find the sum to infinity of:

1. $4 + 1 + \frac{1}{4} + \ldots$

2. $3 - \frac{1}{2} + \frac{1}{12} - \ldots$

3. $\frac{1}{3} + \frac{1}{4} + \frac{3}{16} + \ldots$

4. $-2\frac{2}{3} + 1\frac{1}{3} - \frac{2}{3} + \ldots$

5. $12 - 10 + \frac{25}{3} - \ldots$

***6.** $\frac{128}{3} - 16 + 6 - \ldots$

7. $100 + 10 + 1 + \ldots$

Find the sum of the first four terms and the sum to infinity of each of the following:

8. $\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \ldots$

9. $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots$

10. $0.7 + 0.07 + 0.007 + \ldots$

11. $480 + 0.48 + 0.00048 + \ldots$

Find if possible the sum to infinity of:

12. $\frac{1}{27} + \frac{1}{9} + \frac{1}{3} + \ldots$

13. $\frac{1}{3} + \frac{1}{9} + \frac{1}{27}$

43, C. Repeating Decimals

(See page 160.)

1. How do you know that Formula IV may be used in finding the value of a repeating decimal?

Find the value of each repeating decimal and check by expressing the result in decimal form:

2. $0.444\ldots$

3. $0.2121\ldots$

4. $0.207207\ldots$

5. $4.\dot{8}\dot{4}$

6. $0.23\dot{5}\dot{7}$

7. $70.\dot{7}\dot{0}$

***8.** $0.\dot{4}\dot{8}$

***9.** $0.4\dot{6}$

***10.** $1.6\dot{6}\dot{4}$

11. $8.2\dot{1}\dot{5}$

***12.** $0.\dot{4}8\dot{1}$

***13.** $1.02\dot{6}\dot{3}$

See the tests on pages 164–167.

EXERCISES FOR CHAPTER V

PROBLEM SOLUTION

Exercise 44. Problem Solution

(See pages 169, 180.)

Object: To establish correct habits of problem analysis and attack.

44, A. Preliminary Exercise

1. State and discuss the six steps in the solution of a verbal problem.

2. For each problem, 1–10, of Exercise 44, B. Mention the unknown numbers, describe the relations between the numbers, and tell what plan of solution you expect to use. (Page 170.)

44, B

(See page 184.)

Solve and check the following problems. Use a tabular arrangement of data whenever convenient. Re-read pages 170, 171.

1. The area of the floor of a narrow rectangular sun-porch is 48 sq. ft. The length of a molding which goes completely round it is 32 ft. What are the dimensions of the porch?

2. The area of a triangle is 42 sq. in., and the sum of the base and altitude is 19 in. Find base and altitude.

3. There is just one number in which the sum of the three digits is 16, the units digit is equal to the sum of the other two digits, and interchanging the tens and units digit increases the number by 18. What is that number?

4. If the sides of a certain equilateral triangle are diminished by 1 in., 3 in., and 5 in., respectively, a right triangle is formed. Find the side of the original triangle.

5. Find a number such that if 1, 4, 9, and 16 are added to it in turn, the four resulting numbers will be in proportion.

6. A man invests \$2000 at 4%. How much must he invest at 6% in order to have a yield of 5% on the combined investment?

7. Tickets for a baseball game sold for 25 cents to pupils and 50 cents to others; \$62.50 was received for 150 tickets. How many tickets of each kind were sold?

8. A farmer guarantees his milk to contain $3\frac{1}{2}$% butter fat. How many pounds of 3% milk and of $4\frac{1}{2}$% milk must be mixed to get 80 pounds of $3\frac{1}{2}$% milk?

9. An airplane pursued a departing steamer and ordered it back to port. The steamer had a two-hour start and was overtaken 80 miles from the starting point. When the airplane reached port again, the steamer was still 48 miles at sea. What was the speed of each?

10. Divide 36 into two parts in such a way that one is to the other as 4 is to 5.

11. A workman receives \$180 for a certain job. On his next job he gets \$1 a day more and earns the same amount in six days less time. How much did he get per day on each job?

12. A merchant wishes to buy 100 lb. of tea, part at 48 cents and part at 58 cents a pound, and mix it so that he can sell the mixture at 65 cents a pound and make 25% profit on the transaction. How much of each grade of tea must he buy?

13. A man changes \$50 into shillings and francs and receives

100 shillings and 625 francs. On another occasion he changes \$75 and receives 140 shillings and 1000 francs. Find the exchange rate of each; that is, its value in cents.

***14.** To what will \$2000 amount after eight years at $4\frac{1}{2}\%$ interest compounded semi-annually?

15. A contractor must remove 2200 cubic yards of earth in 30 working days. But one steam shovel can work on the job at a time. His own shovel can remove 65 cubic yards a day, and he can rent for \$60 a day another shovel which will remove 108 cubic yards a day. What is the least rental he must pay? (Parts of a day count as whole days.)

16. To make two tons of 25% sulphuric acid, how much 40% and how much 20% acid would you use?

17. A weight on one end of a lever is just balanced by 100 lb. $7\frac{1}{2}$ ft. from the fulcrum. The 100-lb. weight is to be replaced by a 75-lb. weight. Where must the 75-lb. weight be placed in order to balance?

***18.** A jeweler has three alloys of gold, silver, and lead which are respectively 5 parts gold, 2 silver, 1 lead; 2 parts gold, 5 silver, 1 lead; and 3 parts gold, 1 silver, 4 lead. How many ounces of each shall he take to secure 9 ounces of an alloy that shall contain equal parts of gold, silver, and lead?

***19.** A man invests for his son a certain sum of money in such a way that after two years at $3\frac{1}{2}\%$ the first year and $4\frac{1}{2}\%$ the second year (interest compounded annually), the boy receives \$1297.89. What was the original investment?

***20.** A grocer paid \$22.50 for oranges and sold them at a profit of \$1.05 a box. After paying \$9 for expenses, he had enough of the proceeds left to buy 60 boxes. How many boxes did he buy at first?

44, C. Literal Solutions

(See page 194.)

Object: To develop the ability to make general statements of problems; that is, to think about problems in general terms. This ability is one of the important results of the study of algebra.

1. What fundamental principle for dealing with literal problems is stated in chapter V?

Solve the following literal problems and give a numerical illustration of each one. Check the solutions.

2. Two automobiles start from the same place and travel at the rates of r_1 and r_2 miles an hour respectively. How many miles apart will they be in h hours if they travel in opposite directions? If they travel in the same direction?

3. Tickets to a game were sold to adults for a cents and to children for c cents. The total receipts for t tickets were d dollars. How many tickets of each kind were sold?

***4.** C_1 representatives of one city and C_2 representatives of another city were sent on a trip of investigation which cost d dollars. What was the cost for each representative? For C_1 representatives?

***5.** A store building cost D dollars to build and entails a yearly expense to the owner of d dollars. He rents each of three stores in the building for n dollars a year. What per cent does he receive annually on his investment?

6. A tank capable of holding g gallons is three fourths full of water. How many gallons of an 80% solution of disinfectant must be added to make a solution that will be 6% disinfectant?

***7.** A workman set a fuse to explode a blast of powder in s seconds. He then ran back at the rate of b yards a second. If

sound travels 1086 ft. per second, for how long had he run when he heard the explosion?

***8.** A dealer bought a dozen grapefruit at c cents apiece. He planned to sell them at a profit of 20%. Two of the grapefruit were spoiled, however, and had to be thrown away. What did he have to charge for each of the rest?

***9.** A man has h hours at his disposal. How far can he ride into the country at m miles an hour if he is to walk back at w miles an hour?

***10.** From each corner of a sheet of metal s in. by $2s$ in., small squares $\frac{s}{6}$ in. on a side are cut. What is the volume of the box that is made by turning up the resulting strips?

***11.** A and B together can do a piece of work in t days. A works s times as fast as B. Derive formulas for the number of days each man requires to do the work alone.

EXERCISES FOR CHAPTER VI

SPECIAL PRODUCTS AND FACTORS

Exercise 45. Factoring. Simple Examples

(See page 208.)

45, A. Common Monomial Factor

Factor and check by multiplication:

1. $a^3 - 5a^2 - a$

2. $3a^2 - 3a^3$

3. $5ax^2 - 10a^2x$

4. $4a^4 - 8a^6 + 6a^8$

5. $a^2 - ab - a$

6. $\frac{1}{2}s^2t - \frac{1}{4}st^2 + \frac{3}{4}st$

7. $\frac{1}{3}\pi r^2h + \frac{1}{3}\pi R^2h + \frac{1}{3}\pi Rrh$

8. $a(a^3 + a^2b - a)$

***9.** $x(xy^n - x^2y^{2n})$

10. $na^2 - a^{n+1}$

11. $\frac{4}{3}\pi r^2h - \frac{4}{3}\pi R^2h$

12. $p + prt$

13. $\frac{1}{2}hb_1 + \frac{1}{2}hb_2$

14. $\pi sr + \pi sr'$

15. $\frac{1}{4}x^2yz + \frac{1}{8}xyz^2 - \frac{1}{12}xy^2z$

*16. $x^{\frac{2}{3}}y^{\frac{3}{2}} - x^{\frac{3}{2}}y^{\frac{2}{3}}$

17. $\pi h^2 r - \frac{1}{3}\pi h^3$

*18. $x^{\frac{1}{4}}y^{\frac{3}{4}} - 2x^{\frac{1}{2}}y^{\frac{1}{2}} + x^{\frac{3}{4}}y$

*19. $x^a b - xb^a - xb$

*20. $x^{2a+n} - x^{3a+2n}$

45, B. The General Product of Two Binomials

Factor by grouping in pairs and taking a monomial factor from each pair and combining:

1. $ax - bx + ay - by$
2. $2x^2 + xy + 2xz + yz$
3. $2a^2 - 6a + ax - 3x$
4. $3xy - x + 3y^2 - y$
5. $ax + bx - 2ay - 2by$
6. $a^2 - 4a + ax - 4x$
7. $a^2 - ab - 5a + 5b$
8. $1 - a + b - ab$
9. $ar + bs - rs - ab$
10. $a^5 + a^3 + a^2b + b$
11. $a^2 - a + ab - b$
12. $6ac - 4bc - 8be + 12ae$
13. $8wy + 3xz - 4wz - 6xy$
14. $10ac - 2bc - 15an + 3bn$
15. $w^2x^2 - y^2z^2 - x^2y^2 + w^2z^2$
16. $2 - 2a - a^2 + a^3$
17. $12x^3 - 8xy - 3x^4 + 2x^2y$
18. $15xy^2 - 9y^2z - 35xy + 21yz$
19. $kt - rt + 5k - 5r$
20. $3a^3 + 3a^2 - a - 1$

45, C. Quadratic Trinomials and Binomials, Including the Perfect Square and the Difference of Two Squares

(See Exercises 11 and 13, pages 358, 360.)

45, D. The Recognition of Typical Algebraic Products

Object: The development of skill in the recognition of factorable expressions.

In Exercise 45, E, tell which type-product is illustrated by each example.

45, E. Miscellaneous Factorable Expressions

Factor the following expressions with the help of the five type products which you have committed to memory:

1. $x^2 + 5x + 6$ **2.** $ab + ac + ad$ **3.** $\pi r^2 - \pi r_1^2$

4. $2\pi r^2 + \pi rh$ **5.** $\pi r + \pi r^2$

6. $5x^2 - 10xy + 5x$ **7.** $18a^3y + 24a^4y^2 + 6a^3y$

8. $ax - 3a + bx - 3b$ **9.** $a^2 + ab + ax + bx$

10. $(a - b)(c - d) + 2(c - d)$

11. $(x + 1)(x - 1) + (x + 1)(x - 2)$

12. $a(b - c) + n(c - b)$ **13.** $9 - 10y + y^2$

14. $24m^2n^2 - 47mn - 75$ **15.** $0.09a^2 - 0.24ab + 0.16b^2$

16. $a^3 - 21 + 3a - 7a^2$ **17.** $3x^3 - 1 + 3x^2 - x$

18. $8a^2b^2 - ab^3 - 8a^2 + ab$

19. $4x^2 + 8x + 3$ **20.** $2x^2 + 17x + 30$

21. $4x^2 - 4x - 3$ **22.** $5a^2 - a - 18$

23. $8x^2 + 10xy - 18y^2$ **24.** $3xy - x + 3y^2 - y$

25. $ax + bx - 2ay - 2by$ **26.** $a^2 - 4a + ax - 4x$

27. $a^2 - ab - 5a + 5b$ **28.** $1 - a + b - ab$

29. $ar + bs - rs - ab$ **30.** $a^5 + a^3 + a^2b + b$

***31.** $0.0049x^2 - 0.014x + 0.01$

32. $49m^2 - 144$ **33.** $25a^2b^4 - 1$ **34.** $16a^2x - b^2x$

35. $50a^3 - 25ab^2$ **36.** $\pi R^2 - \pi r^2$ **37.** $a^3 + a^2b + a + b$

38. $6r^2 - rs - s^2$ **39.** $18 - 50m^4$ ***40.** $x^2 + x + \frac{1}{4}$

41. $4x^2 - x - 14$ **42.** $6a^2 - 20a + 6$ **43.** $x^2 - 25x^2z^2$

44. $6ab^3 - 6ab$ **45.** $2x^2 - 2cx - 4dx + 4cd$

46. $7a^0 - 8a + a^2$ **47.** $x^3y^2z^3 + x^2y^2z^4 + x^3yz^4$

48. $7 - 16a + 4a^2$ **49.** $12x^2yz^2 + 6x^2y^2z^3 - 9x^3y^3z$

50. $4 - 12x + 5x^2$ **51.** $2x^2 - 28x + 98$

52. $3a^8 - 768b^{16}$ ***53.** $3b(2m - 3n) - 2a(3n - 2m)$

54. $(a-2)^2 - y^4$ **55.** $x^2y - 5xy - 14y$ **56.** $r^5p - 16rp$

57. $2 - 5a + 2a^2$ ***58.** $\frac{a^2}{4b^2} + \frac{4b^2}{a^2} + 2$

Exercise 46. Factoring. More Complex Factorable Expressions

(See page 212.)

46, A

1. What are some of the uses of factoring?

2. Are $(x - \sqrt{5})$ and $(x + \sqrt{5})$ factors of $x^2 - 5$ according to our definition of a factor? Explain.

3. What are the steps to take in a systematic study of factoring?

4. From the point of view of this chapter, what are the two most important lessons to be learned from a study of factoring?

5. Name five (or six) type products and select from the examples below several illustrations of each type. Tell what complications if any are introduced into each one of them.

6. Give a systematic plan of procedure for factoring algebraic expressions.

46, B. Miscellaneous Exercises in Factoring

1. $(m+n)^2 - 1$ **2.** $9 - (a+b)^2$

3. $a^2 - 2ab + b^2 - c^2$ **4.** $x^2 - 4xy + 4y^2 - z^2$

5. $a^2 - b^2 - 2bc - c^2$

6. $2 - 16x + 32x^2$

7. $25 - a^2 - 9b^2 + 6ab$

8. $12a^2b^2 + 3ab + 15b^2$

***9.** $8x^{2a} - 4x^{4a} - 6x^{3a}$

***10.** $24x^{2n} - 4x^n - 4$

11. $4x^{-2} - 4x^{-1} + 1$

12. $x^{2a} - y^{4a}$

***13.** $x - x^5$

14. $(a+b)^2 + 2(a+b) + 1$

15. $4(x+y)^2 - 12(x+y)(w+z) + 9(w+z)^2$

16. $(a^2 + 4ab + 4b^2) - 2(a+2b) + 1$

17. $x^4 - 3x^2 - 18$

18. $a^{-2} - 2a^{-4} + 4a^{-6}$

19. $25x^4 - 19x^2 + 1$

20. $25a^4 - 11a^2 + 1$

21. $a^4 - a^2b^2 + 16b^4$

***22.** $(a^2 - ab)^2 - (ab - a^2)^2$

23. $3(a+b)^2 - 2(a+b) - 5$

24. $36x^4 - 25x^2 + 4$

***25.** Make a set of factorable expressions, five for each type with which you are familiar, and arrange them in order of their complexity, the simplest ones first.

26. $(a+3)^4 + 32 - 18(a+3)^2$

27. $m^2 - a^2 - m - a$

28. $r^2 - m^2 + s^2 - n^2 - 2rs + 2mn$

***29.** $x^2 + y^2 + max - (2x + ma)y$

***30.** $1 - 2mx - (c - m^2)x^2 + mcx^3$

***31.** $a(a+1)(4a-5) + 6(a+1)$

32. $x^{2a} - 2x^a + 1$

33. $8a - 8a^3b^4$

***34.** $5x^2 + 4.2x + .88$

35. $(m+2)(2m-1) + (m+2)$

***36.** $2ax + 6by - ay - 12bx$

***37.** $x^{-4} - 13x^{-2} + 36$

***38.** $9x^{\frac{1}{2}} - 12x^{\frac{1}{4}} + 4$

***39.** $r^3 - s^3$

***40.** $ax^3 + a$

***41.** $x^6 + y^6$

***42.** $r^6 - 1$

***43.** $x^4 - 8x + x^3 - 8$

*44. $1 - 64\,m^6$

*45. $1 - 729\,a^6$

*46. $(c + d)^3 - 8$

*47. $a^6 - 0.027\,b^6$

48. $(m + n)^2 + 5(m + n) + 6$

49. $4\,a^2 - 4\,ab + b^2 - c^2$

50. $3\,a^{2n} - a^n - 2$

51. $(x^2 - 4)^2 + (x + 2)^2$

52. $4\,x^2 - s^2 + 6\,x - 3\,s$

53. $1 - x + x^4 - x^5$

*54. $4\,p^2 + 52\,p - 192$

55. $9 - 6(a - 2\,b) + a^2 - 4\,ab + 4\,b^2$

*56. $8\,x^3 - 125$

*57. $x^6 - y^6$

*58. $m^6 + 1$

*59. $x^{12} - 1$

*60. $64(a - b)^3 - c^3$

*61. $8\,a^8 - 18\,b^6$

*62. $125\,a^6 - 8\,b^9$

*63. $27(x - y)^3 + 1$

*64. $m^4 - 27\,m + 3\,m^3 - 81$

65. $(2\,a - b)^2 - 4$

*66. $s^5 - s$

*67. $x^{2n} + 2\,x^n y^n + y^{2n}$

68. $36 + 12\,a^{-1} + a^{-2}$

69. $a^{-2} - 8\,a^{-1} + 16$

70. $4\,m^2 - 12\,mn + 9\,n^2 - 4\,r^2$

*71. $2(m^3 - 1) + 7(m^2 - 1)$

72. $9\,a^{-2} - 6\,a^{-1} + 1$

73. $(a + b)^2 - 4(a + b) + 4$

*74. $a^4 + a^3b - ab^3 - b^4$

*75. $x^3 + y^3 + x + y$

76. $3\,x^{-2} - 15\,x^{-1} - 42$

77. $4\,a^4 + 9\,y^4 - 93\,a^2y^2$

*78. $m^4 - mn^3 + m^3n - n^4$

79. $c^2 - 12\,c - 49\,d^2 + 36$

80. $x^2 + x + ax - ab - b - b^2$

*81. $x^3 - 7\,x^2 - 3\,x + 21$

82. $a^4 - 14\,a^2b^2 + b^4$

*83. $x^6 + 1$

84. $3\,a^2 + 13\,a - 10$

*85. $x^4 + x^2 + 1$

*86. $3\,ab(a + b) + a^3 + b^3$

87. $a(a - c) - b(b - c)$

88. $32 - 4\,x - 3\,x^2$

*89. $(y + 1)^3 - 1$

90. $m(m + 1)(m + 3) - 2\,m - 6$

91. $3 - m - 10\,m^2$

*92. $x^4 - \dfrac{y^2}{16}$ *93. $x^{\frac{1}{4}} - z^{\frac{2}{3}}$ *94. $x^2 + xy - 72\,y^2$

*95. $2\,mn + 2\,ln + n^2 + 4\,lm$

*96. $0.2\,m^2 + 0.72\,mn - 1.4\,n^2$

EXERCISES FOR CHAPTER VII

FRACTIONS. (See page 225.)

Exercise 47. Changing the Form of a Simple Fraction

Object: Skill in changing the form of a fraction without changing its value, and in guarding against typical errors.

Multiplication by (-1):

1. Is $\dfrac{a-b}{m-n} \equiv -\dfrac{-a-b}{m-n}$ or $-\dfrac{b-a}{m-n}$?

2. Is $\dfrac{r-s}{x-y} \equiv -\dfrac{r-s}{-x+y}$ or $-\dfrac{r-s}{x-y}$?

3. Is $\dfrac{b-c}{v-w} \equiv \dfrac{-b+c}{-v+w}$ or $\dfrac{c-b}{v-w}$?

Change to one or more equivalent forms with different signs:

4. $\dfrac{-3}{x-6}$

5. $\dfrac{x-y}{y^2-x^2}$

6. $-\dfrac{2\,m+1}{1-m}$

7. $-\dfrac{m-2\,n}{-m-2\,n}$

8. $\dfrac{m^2-2\,m-3}{12-m-m^2}$

*9. $\dfrac{x(a-b)-y(b-a)}{x(b-a)+y(a-b)}$

Other changes in form. Supply the missing expressions:

10. $\dfrac{2}{3\,x} = \dfrac{?}{6\,x}$

11. $\dfrac{3\,y^2}{4\,x^3} = \dfrac{?}{20\,x^4}$

12. $\dfrac{r^3}{5\,n^3} = \dfrac{?}{25\,n^6}$

13. $\dfrac{-3\,a^3}{8\,b^3y} = \dfrac{?}{24\,b^4y^2}$

14. $\dfrac{2\,m}{m-n} = \dfrac{4\,m(m-n)}{?}$

15. $\dfrac{5\,a}{a-b} = \dfrac{?}{(a-b)(a-2\,b)}$

16. $\dfrac{4\,m^2}{(n-m)(r+s)} = \dfrac{?}{(m-n)(r+s)}$

17. $\dfrac{x-y}{4\,a} = \dfrac{(y-x)(x-2\,y)}{?}$

18. $-\dfrac{2\,a}{b^2-a^2} = +\dfrac{?}{(a+b)(a-b)}$

***19.** $\dfrac{x}{x+1} = \dfrac{?}{x^3+1}$

20. $c + \dfrac{b^2-2\,bc+c^2}{2\,b} = \dfrac{?}{2\,b}$

21. $\dfrac{4\,a}{12-a-a^2} = -\dfrac{?}{(a-3)(a+4)}$

22. $\dfrac{1}{z-1} - \dfrac{1}{(1-z)^2} = \dfrac{?}{(z-1)^2}$

23. $\dfrac{1}{\sqrt{2}} = \dfrac{?}{2}$

24. $\dfrac{5\,a}{\sqrt{3\,a}} = \dfrac{?}{3\,a}$

25. $\dfrac{1}{5\sqrt{20}} = \dfrac{?}{\sqrt{5}}$

26. $\dfrac{1}{3-\sqrt{5}} = \dfrac{?}{4}$

27. $\sqrt{\dfrac{3\,a}{5\,b}} = \dfrac{?}{?}\sqrt{?}$

***28.** $\sqrt[6]{\dfrac{4\,a^2}{9\,b^2}} = \sqrt[3]{?}$

29. $2\,x^{-2} = \dfrac{2}{?}$

***30.** $\dfrac{x}{x^{-1}-y^{-2}} = \dfrac{?}{y^2-x}$

***31.** $\dfrac{ab^{-1}-a^{-1}b}{a^{-2}-b^{-1}} = \dfrac{?}{b-a^2}$

32. $\log\dfrac{A}{B} = \log ? - \log ?$

Cancellation, a dangerous word. Insert $\equiv$ or $\not\equiv$ between each pair of expressions. Give reasons for your answers:

33. $\dfrac{x^3+x}{x} \qquad x^3$

34. $\dfrac{x^2-x^6}{x} \qquad \dfrac{1}{x}-x^3$

35. $\dfrac{-4x^4-20x^6+12x^5}{-4x^4} \qquad -1+5x^3-3x$

36. $\dfrac{x^2+2x}{x^2-1} \qquad \dfrac{x+2}{x+1}$

37. $\dfrac{x^2+y^2}{x^2-y^2} \qquad 1$

38. $\dfrac{1+m^3-m^2}{m^4} \qquad \dfrac{1}{m^4}+\dfrac{1}{m}-\dfrac{1}{m^2}$

39. $\dfrac{x^2-6x+9}{x^2-6x+9} \qquad 0$

40. $\dfrac{(-1)(9-6x-x^2)}{x^2-6x-9} \qquad 1$

Which of the following fractions can be reduced to lower terms by "crossing out" the m^2's? Explain:

41. $\dfrac{m^2+t^2}{m^2+s^2}$

42. $\dfrac{m^2t^2}{m^2+s^2}$

43. $\dfrac{s+m^2}{2+3m^2}$

44. $\dfrac{m^2t}{m^2s}$

45. $\dfrac{m^2(r-s)}{3m^2}$

46. $\dfrac{m^2-m^2(a-b)}{m^2}$

47. $\dfrac{m^2\sqrt{mn}}{m^2}$

48. $\dfrac{\sqrt{m^2n}}{m^2}$

49. $\dfrac{m^2\sqrt{ab}}{m^2}$

50. $\sqrt{\dfrac{m^2n}{m^2}}$

51. $\left(\dfrac{am^2}{m^2}\right)^{\frac{1}{2}}$

*52. $\dfrac{(m^2)^{-2}+m^{-2}}{m^2}$

Simplify each of the following fractions which is not already in its simplest form:

53. $\dfrac{a^2-2ab+b^2}{5a-5b}$

54. $\dfrac{a^3-3a^2+2a}{2a^2+10a-12}$

55. $-\dfrac{a+b}{b^2-a^2}$

56. $\dfrac{(\sqrt{3})^2-(\sqrt{2})^2}{(\sqrt{2}-\sqrt{3})^2}$

57. $\dfrac{2x^3 + 5x^2 - 12x}{7x^3 + 25x^2 - 12x}$

58. $\dfrac{a^2 + b^2}{a + b}$

*59. $\dfrac{4a^4 + b^4}{2a^2 + 2ab + b^2}$

*60. $-\dfrac{3x - 3y}{7y - 7x}$

Exercise 48. Addition and Subtraction of Fractions

(See page 226.)

Combine and check:

1. $\frac{5}{6} + \frac{6}{5} + 6$

2. $\frac{5}{8} + \frac{3}{16} + \frac{1}{4}$

3. $\dfrac{2}{0.125} - \dfrac{5.25}{0.375}$

4. $\dfrac{4a + 1}{4} + \dfrac{2a - 1}{3}$

5. $\dfrac{2a - 3b}{8} + \dfrac{a - 2b}{16} + \dfrac{5a + b}{4}$

6. $\dfrac{x - 1}{2x^3} + \dfrac{2x + 1}{3x^2}$

7. $\dfrac{2a + 1}{3a^2} + \dfrac{a - 1}{2a^3}$

8. $\dfrac{b + 1}{b - 1} + \dfrac{b - 1}{b + 1}$

9. $\dfrac{1}{m + 1} + \dfrac{2}{m - 1} - \dfrac{1}{m^2 - 1}$

10. $\dfrac{r}{r - 1} + \dfrac{1}{r^2 - 1} - \dfrac{r}{r + 1}$

11. $\dfrac{3}{a + 1} + \dfrac{3}{a - 1} + \dfrac{3}{a - 2}$

*12. $\dfrac{3}{r} + \dfrac{5}{1 - r} - \dfrac{2r - 3}{r^2 - 1}$

*13. $\dfrac{r}{r - 2} - \dfrac{r - 2}{r + 2} + \dfrac{2}{4 - r^2}$

*14. $\dfrac{4x + y}{x - 2y} + \dfrac{4x - y}{x + 2y} + \dfrac{18xy}{x^2 - 4y^2}$

*15. $\dfrac{a}{a - 1} + \dfrac{a^2}{a^2 + a + 1} + \dfrac{1}{a^3 - 1}$

*16. $\dfrac{1}{a^2 + 7a + 12} + \dfrac{2}{2a + 6} - \dfrac{1}{a + 4}$

*17. $\dfrac{2b}{b-c}+\dfrac{4c}{b+c}-\dfrac{b^2+c^2}{b^2+2bc+c^2}$

*18. $\dfrac{2x}{x^2-1}+\dfrac{1}{x+1}+\dfrac{3}{1-x}$

*19. $\dfrac{x}{x+2}-x-\dfrac{x}{x-2}+1$

*20. $\dfrac{b}{b+1}-\dfrac{1}{b-1}+2$

*21. $\dfrac{a}{\dfrac{1}{b}-\dfrac{1}{c}}+\dfrac{b}{1-\dfrac{b}{c}}-\dfrac{c}{1-\dfrac{c}{b}}$

Express each of the following as a single fraction with rational denominator.

22. $\dfrac{1}{5-\sqrt{3}}+\dfrac{1}{5+\sqrt{3}}$

23. $\dfrac{3}{\sqrt{5}+\sqrt{2}}-\dfrac{\sqrt{10}}{\sqrt{5}-\sqrt{2}}$

24. $\dfrac{\sqrt{3}+3}{2\sqrt{3}-2}+\dfrac{4}{\sqrt{3}-3}$

25. $\dfrac{7-4\sqrt{3}}{2-\sqrt{3}}-\dfrac{1}{2+\sqrt{3}}$

*26. $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}-\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

*27. $\dfrac{\sqrt{7}}{\sqrt{7}-4}-\dfrac{2}{2\sqrt{7}+3}$

*28. $\dfrac{\sqrt{3}-\sqrt{2}}{2\sqrt{6}-\sqrt{2}}-\dfrac{9}{\sqrt{6}}$

*29. $\dfrac{1}{\sqrt{x}+\sqrt{y}}-\dfrac{1}{\sqrt{x}-\sqrt{y}}$

Exercise 49. Multiplication and Division of Fractions

(See page 226.)

1. $\frac{4}{5}\times\frac{15}{6}$ $\qquad 5\frac{1}{3}\times 2\frac{1}{4}$ $\qquad 5\div\frac{1}{8}$ $\qquad \dfrac{2\frac{2}{3}}{4\frac{1}{8}}$

2. $\dfrac{x^2y^2z^2}{abc}\times\dfrac{xy^2z^3}{a^3bc^2}\div\dfrac{x^2y^3z}{ab^2c^4}$

3. $\dfrac{x^2+xy}{x^2-y^2}\div\left(\dfrac{x}{x-y}-\dfrac{y}{x+y}\right)$

4. $\dfrac{z(x-y)}{ms}\times\dfrac{z(y-z)}{m^2s^{-1}}\div\dfrac{x^2-y^2}{m^3}$

5. $\dfrac{6a^3b^{\frac{2}{3}}c^{n+1}}{x^2yz^3}\times\dfrac{-3a^4b^{-\frac{2}{3}}c^{n-3}}{x^0y^{-1}z^{-2}}$

*6. $\left\{\sqrt{\dfrac{a^{\frac{1}{2}}x^{-2}}{x^{\frac{1}{2}}a^{-2}}}\times\sqrt{\dfrac{a\sqrt{x}}{x^{-1}\sqrt{a}}}\right\}$

Perform the operations indicated:

7. $\dfrac{3}{a-b} \div \dfrac{9}{a^2-b^2}$

8. $\dfrac{a+b}{a-b} + \dfrac{a-b}{a+b} \times \dfrac{a-b}{a^2+b^2}$

9. $\dfrac{5a}{3-x} \times \dfrac{a}{x-3}$

10. $\dfrac{m+1}{m+2} \cdot \dfrac{m+2}{m+3}$

11. $\left(\dfrac{1}{a-b} - \dfrac{1}{a+b} - \dfrac{1}{b^2-a^2}\right) \div \dfrac{2a+1}{a-b}$

12. $\dfrac{x+2}{x+5} \cdot \dfrac{x^2-25}{x^2-4} \cdot \dfrac{x^2-3x+2}{x^2-6x+5}$

13. $\dfrac{x^2y+xy^2}{x^2y-xy^2} \cdot \dfrac{x^3-xy^2}{x^3+xy^2}$

14. $\dfrac{3a^2+a-2}{4a^2-4a-3} \div \dfrac{2a^2-a-3}{6a^2-a-2}$

15. $\left(\dfrac{a}{b} - \dfrac{b}{a}\right) \div \dfrac{a+b}{ab}$

***16.** $\left(\dfrac{a+b}{a-b} + \dfrac{a-b}{a+b}\right) \div \left(\dfrac{a+b}{a-b} - \dfrac{a-b}{a+b}\right)$

***17.** $\dfrac{1}{ab}\left(\dfrac{a}{b} - \dfrac{b}{a}\right) - \dfrac{a^3+b^3}{a^3b^3}$

***18.** $\dfrac{x^6-y^6}{2xy} \times \dfrac{4x^2y^2(x+y)}{(x^2+xy+y^2)(x^2-xy+y^2)} \div \dfrac{(x+y)^3}{2x^2y-2xy^2}$

***19.** $\left(a + \dfrac{b^2}{a-b}\right) \div \left(a - \dfrac{b^2(a-b)}{a^2+b^2}\right)$

***20.** $\left(\dfrac{r+s}{r-s} - \dfrac{r-s}{r+s}\right)\left(\dfrac{2r}{r-s} + r + s\right) \div \left(\dfrac{r-s}{r+s} + \dfrac{r+s}{r-s}\right)$

***21.** $\dfrac{a+b}{a^2+ab+b^2} \cdot \dfrac{a^2-ab+b^2}{a-b} \cdot \dfrac{a^3-b^3}{a^3+b^3}$

***22.** $\left(\dfrac{x-y}{x+y} + \dfrac{x+y}{x-y}\right) \div \left(\dfrac{x^2-y^2}{(x-y)^2} - 1\right)$

***23.** $\dfrac{a-b+1}{a+b-1} \div \left(\dfrac{a^2-(b+1)^2}{b^2-(a-1)^2} \cdot \dfrac{1-(a-b)^2}{(a+b)^2-1}\right)$

***24.** $\dfrac{1}{1-(1+a)^{-1}}+\dfrac{1}{1-(1-a)^{-1}}$

***25.** $\dfrac{1}{x-\sqrt{x^2-4}}+\dfrac{1}{x+\sqrt{x^2-4}}$

***26.** $\dfrac{a^2b^{-1}+b-a}{b^{-1}-a^{-1}}\div\dfrac{a^3+b^3}{a^2-b^2}$

***27.** $\dfrac{2x^{\frac{1}{2}}-y^{\frac{1}{3}}}{x^2-y^2}\cdot\dfrac{x^{-1}-y^{-1}}{x^{\frac{1}{2}}y^{\frac{1}{3}}}\div\dfrac{2x+3x^{\frac{1}{2}}y^{\frac{1}{3}}-2y^{\frac{2}{3}}}{x^{\frac{1}{2}}+2y^{\frac{1}{3}}}$

Exercise 50. Complex Fractions

(See page 228.)

Simplify:

1. $\dfrac{\frac{1}{3}}{5}$ **2.** $\dfrac{\frac{2}{3}}{\frac{1}{2}}$ **3.** $\dfrac{\frac{5}{8}}{2\frac{3}{4}}$ **4.** $\dfrac{\frac{4}{5}}{\frac{3}{7}}$

Simplify, and check by numerical substitution:

5. $\dfrac{\dfrac{a+b}{c}}{\dfrac{a-b}{c}}$ **6.** $\dfrac{\dfrac{a+b}{ab}}{\dfrac{a-b}{c}}$ **7.** $\dfrac{a-\dfrac{a-b}{a}}{1+\dfrac{a+b}{ab}}$

***8.** $\dfrac{\dfrac{c}{d^2}+\dfrac{d}{c^2}}{\dfrac{1}{c^2}-\dfrac{1}{cd}+\dfrac{1}{d^2}}$ **9.** $\dfrac{\dfrac{a}{b}+\dfrac{b}{a}}{a-b}\times\dfrac{\dfrac{b}{a}-\dfrac{a}{b}}{a^2+b^2}$

***10.** $\dfrac{\dfrac{1}{a-2}-\dfrac{1}{a-3}}{\dfrac{1}{a^2-5a+6}+1}$ **11.** $\dfrac{\dfrac{a}{a-b}}{\dfrac{a^2}{a^2-b^2}}$

12. $$\frac{\dfrac{r}{r-a}-1}{1-\dfrac{r}{r+a}}$$

*13. $$\frac{\dfrac{x}{1+x}+\dfrac{1-x}{x}}{\dfrac{x}{1+x}-\dfrac{1-x}{x}}$$

*14. $$\frac{\dfrac{m+1}{m}-\dfrac{4}{m+1}}{\dfrac{1}{m+1}-\dfrac{1}{m}}$$

*15. $$\frac{x-\dfrac{xy}{x+y}}{x+\dfrac{xy}{x+y}}$$

*16. $$\frac{a-5+\dfrac{6}{a+2}}{a-3-\dfrac{5}{a+1}}$$

*17. $$\frac{\dfrac{1}{x}-\dfrac{x+y}{x^2+y^2}}{\dfrac{1}{y}-\dfrac{x+y}{x^2+y^2}}$$

*18. $$\frac{\dfrac{b(a+b)}{a}\left(\dfrac{x}{a}-\dfrac{x}{b}\right)}{a-\dfrac{b}{a}}$$

19. $$\frac{\dfrac{a}{\frac{b}{c}}}{g};\quad \frac{\dfrac{a}{b}}{\dfrac{c}{g}}$$

20. Write the reciprocal of each of the following numbers:

$\frac{a}{b}$ $\quad \frac{1}{2}$ $\quad 7$ $\quad \frac{a-b}{c}$ $\quad 1$ $\quad -5$ $\quad 4\frac{2}{3}$

Simplify:

21. $1+\frac{1}{2}$ $\qquad 1-\dfrac{1}{1+\frac{1}{2}}$ $\qquad 1-\dfrac{1}{1-\dfrac{1}{1+\frac{1}{2}}}$

*22. $$1-\frac{1}{1+\dfrac{2}{1-\frac{2}{3}}}$$

*23. $$1+\frac{1}{1-\dfrac{1}{1+\dfrac{1}{b}}}$$

*24. $$\frac{a^3-\dfrac{8}{b^3}}{a^3b^3-a^2b^2}\times\frac{\dfrac{1}{ab}+\dfrac{1}{a^2b^2}}{1+\dfrac{2}{ab}\times\dfrac{4}{a^2b^2}}\div\frac{ab+1}{ab-1}$$

(See the tests on pages 228, 229.)

Exercise 51. Proportion

51, A. (See page 231.)

Solve for x each proportion below and check your solution:

1. $\frac{x}{3} = \frac{8}{3}$ 2. $\frac{1}{5} = \frac{3}{2x}$ 3. $\frac{1}{a} = \frac{1}{x-a}$

4. $\frac{4}{7x+3} = \frac{3}{6x+2}$ 5. $\frac{5}{3x+2} = \frac{7}{5x-2}$

6. $\frac{x-3}{x+5} = \frac{-1-x}{2-x}$ 7. $\frac{5}{x-2} = \frac{8}{x^2-4}$

8. $\frac{a}{3} = \frac{3}{ax}$ 9. $\frac{1}{x-a} = \frac{3}{b+c}$ 10. $\frac{1}{r-x} = \frac{3}{s-x}$

11. $\frac{a-x}{b-x} = \frac{a-1}{b-1}$ 12. $\frac{a-b}{x} = \frac{x}{a-b}$

51, B

Transform each proportion below into the form: $\frac{a}{b} = \frac{c}{d}$ or $\frac{a}{b} = \frac{b}{c}$

1. $\frac{a-2b}{b} = \frac{c-2d}{d}$ 2. $\frac{ab}{cd} = \frac{b^2}{d^2}$ 3. $\frac{\sqrt{a^2-b^2}}{\sqrt{c^2-d^2}} = \frac{b}{d}$

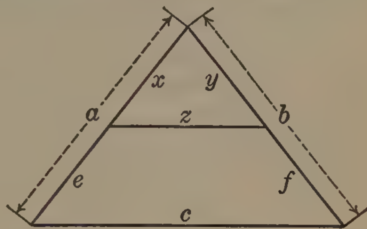
4. $\frac{a^2-b^2}{b^2} = \frac{c^2-d^2}{d^2}$ 5. $\frac{a^2-6b^2}{b^2} = \frac{c^2-6d^2}{d^2}$

*6. $\frac{a^2d^2e^2 - 2abcde^2 + b^2c^2e^2}{b^2d^2} = 0$ 7. $b = \pm\sqrt{ac}$

8. $\frac{a-b}{b-c} = \frac{b}{c}$ *9. $\frac{a^2-b^2}{2ab} = \frac{b^2-c^2}{2bc}$

*10. In similar figures corresponding lines are proportional.

For example, in the accompanying figure, $\frac{a}{b} = \frac{x}{y}$ $\frac{a}{x} = \frac{c}{z}$ $\frac{c}{z} = \frac{b}{?}$ $\frac{b}{c} = \frac{y}{?}$ etc. Show that if $\frac{a}{x} = \frac{b}{y}$, then $\frac{e}{x} = \frac{f}{y}$.



Plan: Substitute $(a - x)$ for e, etc.

*11. In the figure of the preceding example, if $a = 20$, $b = 25$, $x = 15$, and $z = 10$, find the lengths of the other lines. Express the answers in fractional form.

51, C. Problems. (See page 234.)

Solve by proportion or by equations in the "k form." (Mathematicians often prefer the "k form.")

1. The areas of similar figures are proportional to the squares of corresponding lines; that is, $\frac{A}{A_1} = \frac{s^2}{s_1^{\,2}}$ or $\left(\frac{s}{s_1}\right)^2$; also $A = ks^2$. The side of one polygon is 8 in. and the area 20 sq. in. The area of a similar polygon is 180 sq. in. Find the corresponding side. Supply the missing numbers:

s	8	?	?	16	20	
A	20	180	500			2000

*2. The sides of a polygon are 9.9, 9.3, 4.5, 8.6, and 4.9 in. respectively. Find the sides of a similar polygon of which the longest side is 24 in. $s_1 = ks$. Find k to 3-figure accuracy and find the sides to 2-figure accuracy. Prove your work by showing that one perimeter is k times the other.

*3. On a small map the distance A to B is 3.1 in., C to D is 4.8 in., E to F is 7.3 in., and G to H is 10.3 in. In making a

large copy of this map the distance from A to B is 15 in. How far should it be from C to D? From E to F? From G to H? Work as suggested in the preceding problem.

4. The pressure of the wind on any surface varies as the square of the velocity of the wind. On a certain surface the pressure was 200 lb. when the wind blew at a velocity of 15 miles an hour. Supply the missing numbers:

v	15	30	40	50	60	100	
P	200						500

5. The formula for the area of a circle is $A = \pi r^2$. The relation between the areas of two circles is given by the proportion $\frac{A}{A_1} = \frac{?}{?}$ Show how to develop this proportion from the formula. The areas of two circles are proportional to the . . . of their . . The radius of a circle is 13 in. and that of another circle 6 in. How do their areas compare? The area of a circular grass plot is 100 sq. ft., and that of another circular grass plot is 225 sq. ft. What is the ratio of their radii?

6. Starting with the formula $c = 2\pi r$, for the circumference of a circle, develop a proportion and make an equivalent statement using the word proportional, as was done in the preceding exercise. Make and solve at least two problems which can be solved by the aid of your proportion.

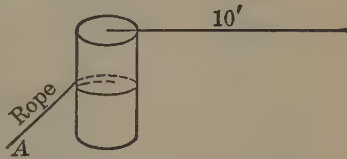
***7.** The law of the lever may be stated $dw = d_1w_1$. (See page 50.) Write this law in the form of an inverse proportion. If one of a pair of horses pulls 450 lb. and the other pulls 600 lb., and if the horses are attached one at each end of 40-inch "evener," at what point on the evener should the load be attached?

8. A "see-saw" is 11 ft. long and balances in the middle. Two children weighing 60 lb. and 55 lb. respectively want to balance each other. If the 55-lb. child sits at the end of his

half of the see-saw, how far from his end must the other child sit?

9. A beam 20 ft. long has a 700-lb. weight on one end and a 300-lb. weight on the other. At what point will the beam balance? (Disregard the weight of the beam.)

10. A horse pulls 400 lb. on the end of the 10-ft. arm of a windlass. The radius of the drum of the windlass is 9 in. What pull is exerted on the rope A?



11. Assuming that the power required to drive a motor-boat is proportional to the square of the speed, how much power will it take to drive at 12 miles an hour a boat which a 5-horse-power motor drives at 8 miles an hour?

12. The interest on a sum of money at a given rate is proportional to the time. If a certain sum bears $165 interest in 3 months, how much will it bear in 11 months?

***13.** The intensity of the light from a given source upon any object is inversely proportional to the square of the distance between them. A book which was 2 ft. from a lamp was moved until it received only one third as much light as at first. How far from the lamp was it moved?

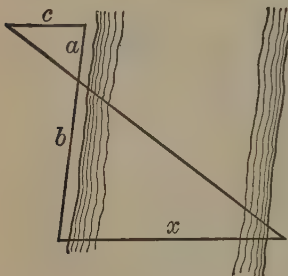
***14.** A book held 8 ft. from a lamp is insufficiently illuminated. How near the lamp must it be held in order to double the intensity of illumination?

***15.** A certain desk top is 8 ft. from a ceiling lamp and is illuminated correctly by a 100-candle-power lamp. If the lamp is lowered 4 ft., what candle-power will give the same illumination on the desk?

***16.** A photographic print is printed by a 10-second exposure

at a distance of 12 in. from a lamp. How long should it be exposed if placed 20 in. from the lamp?

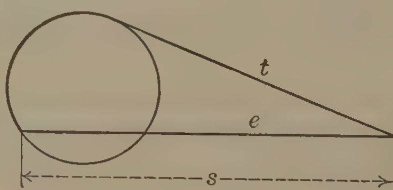
17. Find the height of a tree which casts a 140-ft. shadow when a pole 5 ft. 8 in. high casts a 7-ft. shadow.



18. Find to two-figure accuracy the distance, x, across the river, when $a = 30$ ft., $b = 80$ ft., and $c = 40$ ft.

19. How long must a rectangular rug 9 ft. wide be made so that when it is divided crossways into four small rugs, each small rug will have the same shape as the large rug? (*Plan:* Represent on a drawing the numbers involved; set up the proportion implied in the words "same shape.")

20. In the accompanying figure, t is a mean proportional between s and e.

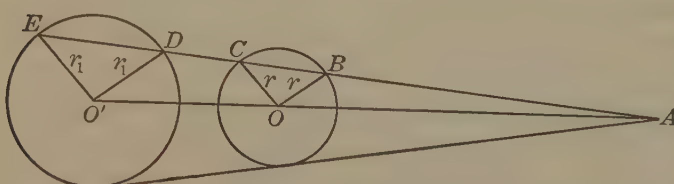


Find t if $s = 8\frac{1}{3}$, and $e = 3$.

Find e if $t = 9$, and $s = 27$.

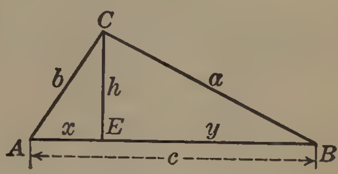
Find e if $s = 9$, and $t = 6$.

***21.** If triangles AOB and $AO'D$ are similar, and if triangles



AOC and $AO'E$ are similar, prove that $AE \cdot AB = AD \cdot AC$. (Let $AB = w$, $AC = x$, etc.)

*22. If triangles ACE, BCE, and ACB are similar, prove that:



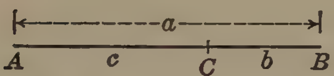
(1) $\dfrac{x}{b} = \dfrac{h}{a}$

(2) $\dfrac{a}{y} = \dfrac{b}{h}$

(3) $a^2 = cy$

(4) $b^2 = cx$

By adding (3) and (4) prove that (5) $a^2 = b^2 = c^2$



23. If $\dfrac{a}{c} = \dfrac{c}{b}$ and $a = 15.0$ in., find c and b to three-figure accuracy.

*24. The scale of a map is 1 in. = 50 mi. How far apart are two places which are $2\frac{5}{32}$ in. apart on this map?

Exercise 52. Fractional Equations

(See page 235.)

Solve and check:

1. $\dfrac{x^2+7}{x+3} - (x-7) = 3$

2. $\dfrac{6\,x-3}{2\,x+7} = \dfrac{3\,x-2}{x+5}$

3. $\dfrac{6\,x+8}{2\,x+1} - \dfrac{2\,x+38}{x+12} = 1$

4. $\dfrac{x+2}{x-3} + \dfrac{2\,x+4}{2\,x-1} = 2$

5. $\dfrac{2}{1-2\,x} + \dfrac{4}{1-4\,x} = \dfrac{6}{1-3\,x}$

6. $\dfrac{3\,x-1}{x+3} - \dfrac{2\,x+3}{1-x} = \dfrac{5\,x^2-2\,x-1}{x^2+2\,x-3}$

*7. $\dfrac{2\,x+5}{x^2+9\,x+14} = \dfrac{x-1}{x^2-x-6} - \dfrac{5-x}{x^2+4\,x-21}$

8. $\dfrac{3}{3-x} - \dfrac{8\,x+3}{9-x^2} = -\dfrac{4}{3+x}$

9. $\dfrac{9}{5x} - \dfrac{8}{10x-5} = \dfrac{4x-1}{4x^2-1}$

10. $\frac{3}{5}(2x-7) - \frac{2}{3}(x-8) = \dfrac{4x+1}{15} + 4$

11. $\dfrac{8x-5}{10} - \frac{3}{2}(x-8) = 6$

12. $\dfrac{2x-3}{x-6} - \dfrac{2-3x}{7} - x = \dfrac{15-12x}{21} - 2$

*13. $\dfrac{x+1}{x^2-2x} - \dfrac{3x+2}{x^2+x} + \dfrac{2x-1}{x^2-x-2} = 0$

14. $\dfrac{2x-1}{2x-2} - \dfrac{23}{10x-10} = \dfrac{3}{5}\left(\dfrac{1}{x-1} - \dfrac{1}{3}\right)$

15. $\dfrac{x}{3} + \dfrac{y}{4} = 6$
$\dfrac{x}{4} + \dfrac{y}{2} = 7$

16. $\dfrac{13+x}{7} + \dfrac{3x-8y}{3} = x + y - \frac{16}{3}$
$\dfrac{11-x}{2} + \dfrac{4x+8y-2}{9} = 7$

17. $\dfrac{7x+8}{5} - \dfrac{7y-1}{4} = -2$
$\dfrac{2x-4}{2} + \dfrac{y-1}{3} = -\frac{1}{3}$

18. $\dfrac{2}{x-1} - 1 = \dfrac{3x-1}{x+3}$

19. $\dfrac{x+3}{3} - \dfrac{7}{3} = \dfrac{3x-1}{2(x+1)}$

20. $\dfrac{5}{2x+3} - \dfrac{8x^2-13x-64}{6x^2+x-12} = \dfrac{7}{4-3x}$

Express the answers to 22–24 in radical form:

21. $\dfrac{2x}{x^2-3x+2} + \dfrac{5}{x-2} = 1$

22. $\dfrac{x+1}{x} - 1 + \dfrac{x}{x+1} = 0$

23. $\dfrac{x+1}{x-2}+\dfrac{2x+1}{x+1}=\dfrac{3x+3}{x-1}$

*24. $\dfrac{x^2-x-4}{x-4}-x-2=\dfrac{1}{4-x}+\dfrac{16}{x^2-16}$ Find the roots correct to nearest hundredth.

Solve for x and check:

25. $x-1-\dfrac{x-2}{a}=\dfrac{1}{a^2}$

26. $\dfrac{x-m}{x}+\dfrac{2x}{x-m}=3$

*27. $2a\left(\dfrac{a}{2}+\dfrac{bx}{a}\right)=4\left(b^2+\dfrac{a^2}{4}\right)$

28. $\dfrac{x}{a}+\dfrac{y}{2}=5 \qquad \dfrac{x}{2a}+y=3$

29. $x+\dfrac{c}{a+b}=\dfrac{cx}{a^2-b^2}+a-b$

*30. $\dfrac{mx-b}{mx+b}-\dfrac{bx-m}{bx+m}=\dfrac{m-b}{(mx+b)(bx+m)}$

31. $\dfrac{x-a}{x-b}=\dfrac{a}{b}$

32. $\dfrac{c-dx}{cx-d}=\dfrac{c}{d}$

*33. $\dfrac{mx^2}{n-px}+m+\dfrac{mx}{p}=0$

34. $\dfrac{a-x}{a+x}=\dfrac{x}{a-x}$

35. $\dfrac{ax-b}{a}-\dfrac{bx+c}{a}=abc$

36. $\dfrac{2c}{a}+\dfrac{b}{x}=\dfrac{c}{2-x}-\dfrac{2cx}{a(2-x)}$

See the problems on pages 236, 237 and the tests on pages 238–241.

EXERCISES FOR CHAPTER VIII

DEPENDENCE

Exercise 53. Graphs

53, A. Understanding Simple Graphs

(See page 247.)

Answer the questions relating to the graphs in the following examples:

1. Page 243, Number 4. **2.** Page 244, Number 5.

3. Page 245, Number 6. **4.** Page 246, Number 7.

53, B. Selecting Suitable Scales

For graphs which are to be drawn on 5 in. by 8 in. paper, with suitable margins, select, by the method of trial and error scales for representing:

1. 18 units horizontally and 30 units vertically.

2. 150 units horizontally and 200 units vertically.

3. 2000 units horizontally and 3500 units vertically.

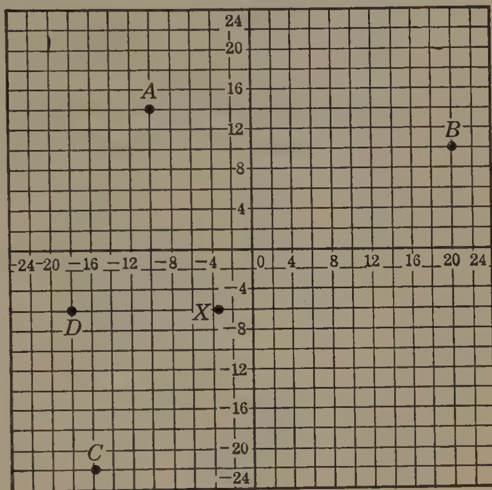
4. From -8 to 20 units horizontally and from -30 to 15 units vertically.

***5.** Select scales for exercises 4–7 if 8 in. by 10 in. paper ruled in $\frac{1}{4}$ in. squares is used.

6. Deposits in a certain bank were as follows (in thousands of dollars): first year, 18; second, 22; third, 29; fourth, 35; fifth, 36. Represent these facts on a bar graph similar to that of Exercise 7, page 246. Select the scale of dollars in such a way as to make the growth look large, as the bank might do were it to use the graph for advertising purposes.

53, C. Plotting Points; Cartesian Scale

1. The point A on the accompanying chart is designated $(-10, 14)$. Its abscissa, x, is -10, and its ordinate, y, is 14. (The abscissa should be mentioned first.) In the same way designate points B, C, D. Point $(-3.8, -6)$ is indicated by X.



Find the locations of:

2. $(-10, 14)$ $(-8, 13)$ $(-13, 8)$ $(8, 13)$ $(-8, -13)$

3. $(5, 0)$ $(0, 5)$ $(0, 0)$ $(2.5, 3)$ $(2.8, -4.2)$

4. Show that a signed number such as -4 or $+9$ represents both a distance and one of two opposite directions.

53, D. Plotting Graphs by Points; Cartesian Coördinates

(See page 248.)

1. Draw a graph to exhibit the fact that the ratio of 1 kilogram to 1 lb. is approximately 2.2. Extend it to 10 kilograms. Compare your work with the graph on page 243 and make numerical exercises and solve them by means of your graph.

2. Draw a graph which may be used to find the approximate number of dollars in a given number of English pounds. 1 £ = \$4.86 approximately. Extend the graph to £ 100.

3. Soundings show the following depths in a river, measured from the left bank toward the right bank:

Distance from left bank..	10′	20′	30′	40′	50′	60′	70′
Depth................	2′	8′	1′	9′	14′	19′	22′

Draw a graphical representation of the bottom of the river. Compare your work with the graph on page 244. Answer questions similar to the ones asked there.

4. Represent graphically the facts in the following table:

Year.........	1820	1840	1860	1880	1900	1920
Number of depositors in savings bank	8,630	78,700	639,900	2,335,600	6,107,000	11,427,600

5. Plot a graph of the formula $5\ a = \frac{1}{2}(2\ b - 3)$. Compare your work with the graph of Exercise 6, page 245. Make a table first.

6. In the formula $a = \dfrac{3\ b^2 c}{d}$ if $b = 2$ and $c = 7$, plot a graph showing the dependence of d on a.

7. On which graphs, 1–6, is it possible to extrapolate as was done in the graph of study 4, page 243.

8. Plot on the same axes graphs of:
$x = y$, from $y = -10$ to 10; $x = y^2$ from $y = -5$ to 5; and $x = y^3$, from $y = -3$ to 3.

***9.** In the interest formula $i = prt$, if $p = \$500$ and $r = 6\%$, plot a graph showing the dependence of i on t. Extend the graph from $t = 1$ year to $t = 20$ years. (In the compound interest formula $A = p(1 + r)^t$, if $p = \$1$ and $r = 6\%$, plot the graph to represent the dependence of A upon t. Extend the graph from $t = 1$ to $t = 8$. In making the computations, use logarithms if convenient.)

***10.** (*a*) In the simple interest formula $A = p(1 + rt)$, if $p = \$1.00$ and $r = 6\%$, plot a graph to show the dependence of A on t between $t = 0$ and $t = 8$. (*b*) In the compound interest

formula $A = p(1 + r)^t$, if $P = \$1$ and $r = 6\%$, plot a graph to show the dependence of A on t between $t = 0$ and $t = 8$. Use logarithms when convenient. What is there in the rates of growth at simple and at compound interest which explains the contrast in these two graphs?

***11.** Look up the derivation and meaning of the word *locus* and tell why the graph of an equation may be referred to as the "locus of the equation."

53, E. Graphic Solution of Pairs of Linear Equations

(See page 255.)

1. What is the locus of the equation $x = 2$, that is, of the equation $x + 0\,y = 2$? Of the equation $y = 2$?

2. Tell how to draw the graph of a linear equation. Tell how to locate the intercepts.

3. Plot the graph of the equation $\dfrac{x+9}{3} = y$. On the graph locate the root of the equation $\dfrac{x+9}{3} = 0$.

4. Tell how to solve graphically a pair of linear equations in two unknowns. Why is this method called the method of intersecting loci?

5. How do graphs show that a pair of equations are dependent? That they are inconsistent?

Solve graphically or tell why you cannot do so:

6. $x + y = 4 \qquad x - y = 2$ **7.** $3\,x = 2\,y \qquad x + 4\,y = 14$

8. $x = 3\,y \qquad x = 3\,y + 5$ **9.** $2\,x - y = 3 \qquad 4\,x = 2\,y - 6$

10. $x - 0\,y = 5 \qquad y = 6$ **11.** $5\,x - y = 16 \qquad x = y$

12. $x - y = 2 \qquad 2\,x + y = 6$ **13.** $y - x = 4 \qquad x = 5 - y$

14. $x+2y=7 \qquad 2x+4y=14$

*15. $6x+\frac{1}{2}y=3\frac{1}{6} \qquad \frac{1}{3}x-3y=-\frac{5}{6}$

*16. $x=2+y \qquad y-x=-4 \qquad y=3x$

53, F. Graphs of Quadratic Equations

(See page 259.)

Plot the graph of each equation below. Comment on the points for which $y=0$; that is, the point at which the graph crosses the x-axis. If these points represent roots of equations, verify the roots.

1. In plotting the graph of a linear equation, it is necessary to locate how many points? It is desirable to locate how many points? In plotting the graph of a quadratic equation, how do you *explore* a portion of the curve which you have not yet definitely located? How do you discover the regions into which a given graph cannot extend? *How do you distinguish between a parabola and a hyperbola?

2. $y=x^2+x-6$. For what values of x is y positive? Negative? Zero?

3. $y=6x^2-x-2$ 4. $y=2x^2-x$ 5. $y^2=2x$

6. $x^2+y^2=20$ 7. $x^2+9y^2=81$ 8. $x^2-y^2=9$

9. $xy=8$ 10. $(x-5y)(2x+3y)=0$

*11. $y=2x^2-3x-9$ *12. $y=2x^2+6x-3$

*13. $3y=2x^2$ *14. $x^2+y^2=25$

*15. $4x^2+4y^2=49$ *16. $2x^2+y^2=10$

*17. $x^2-2y^2=16$ 18. $x=\dfrac{6}{y}$ 19. $2x^2+4y^2=9xy$

20. Solve the following equation graphically by setting the quadratic expression in x equal to y, plotting the graph, and

reading the coördinates of the intercepts with the x-axis: $3x^2 - 5x + 2 = 0$.

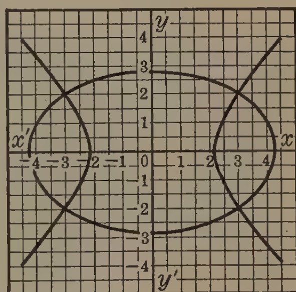
21. Solve as in the preceding example, estimating answers to nearest tenth: $x^2 + 11 = 7x$.

22. Try to solve $x^2 - 4x + 30 = 0$ by the same method. Explain the result.

53, G. Graphic Solution of Pairs of Equations Involving Quadratics

(See page 261.)

1. Read from the graph the roots of the pair of equations, $4x^2 + 9y^2 = 72$ and $x^2 - y^2 = 5$. Check the roots by substitution.



Solve graphically:

2. $x = 2y^2 + 1$
$x - y = 2$

3. $xy = 6$
$3x - 2y = 5$

***4.** $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$
$\dfrac{x}{5} + \dfrac{y}{3} = 1$

***5.** $x^2 + y^2 = 4$
$y^2 = x - 2$

***6.** $x^2 - y^2 = 9$
$y^2 - x = 6$

***7.** $2x - 3y = 6$
$2y + 1 = 4x - 4x^2$

***8.** $x^2 + y^2 = 25$
$xy = 12$

***9.** $x^2 - 9y^2 = 0$
$x - 2 = y^2$

***10.** Solve graphically the equation $x^3 - 4x + x^2 - 4 = 0$. There are three roots. Use the method of example 20, page 452. Solve graphically the pair of equations $x^3 - 4x + x^2 - 4 = y$ and $7x - 1 = y$.

53, H. Statistical Graphs

(See page 267.)

1. Study the graphs on pages 265, 266 and answer the questions asked about them there.

Represent the following relations graphically. If the kind of graph is not specified, use the kind that seems to you most appropriate.

2. The following table gives comparative prices for the period during and immediately after the World War. Draw both graphs on the same axes. When was the farmer prosperous and when comparatively poor?

	1913	'14	'15	'16	'17	'18	'19	'20	'21	'22
Farm crops......	100	108	110	124	208	224	234	238	109	113
Other commodities	100	94	97	132	176	186	195	234	161	163

3. An investigation made some time ago showed that it took an American laborer 3 days to earn enough to pay for a woolen suit. At the same time it took a French laborer 5.2 days, an Englishman 7.7 days, and a German 9.2 days.

4. A railway train went 50 miles an hour for 30 minutes; 65 miles an hour for 30 minutes; 0 miles an hour for 15 minutes; 30 miles an hour for 60 minutes; and 20 miles an hour for 15 minutes.

***5.** Approximate areas in square miles: Georgia, 60,000; Kentucky, 40,000; North Dakota, 70,000; Washington, 70,000; Utah, 85,000.

***6.** A city school department spent $240,000 for salaries, $105,000 for buildings, and $55,000 for other expenses.

***7.** For each 100 employees in a certain industry, 9 worked over 60 hours a week, 60 worked 55–60 hours a week, 21 worked 45–55 hours, and the rest less than 45 hours.

***8.** In a recent year the United States Government paid 23% of its income for interest, 11% to reduce its indebtedness, 10% for the Veterans' Bureau, 17% for the army and navy, 5% for pensions, and the rest for other expenses.

***9.** In a campaign for a safe and sane Fourth of July, the following facts were reported:

	Fatal Accidents	Injuries
First year.......	466	3983
Second.........	183	3986
Third...........	182	4994
Fourth.........	158	5308
Fifth...........	164	4249
Sixth...........	163	5460
Seventh.........	215	5092
Eighth..........	131	2792
Ninth..........	57	1546
Tenth..........	41	947
Eleventh........	32	1131

10. Of each 1000 pupils who finish grade 4, 634 enter grade 8, 342 enter grade 9, 139 finish high school, 72 enter college, and 23 are graduated.

***11.** Per capita expenditure for public schools:

Year.........	1876	1886	1896	1906	1916	1926
Expenditure...	$1.85	$1.97	$2.62	$3.66	$6.26	$16.25

Exercise 54. Ratio, Proportion, and Variation

(See page 253. See also Exercise 51, A.)

54, A. Expressing Proportional Variation in Various Ways

In each of the following illustrations of proportional variation, the nature of the dependence may be expressed (1) by a formula in the k form, (2) by a proportion, (3) by the word *proportional*, (4) by the word *ratio*, and (5) by the words *vary as*; state several of the variations in all of these ways.

Some common laws of variation:

1. x varies directly as y.

2. x varies inversely as y.

3. x varies as the square root of y.

4. y varies inversely as the cube of x.

5. y varies as the square root of x.

6. x varies directly as y and inversely as z.

7. a varies as the cube of b and inversely as the square root of c.

From Geometry and in the Natural Sciences

8. In similar figures,† that is, figures having the same shape, but not necessarily the same size —

(*a*) The ratio of any two corresponding lines is equal to the ratio of any other two corresponding lines.

(*b*) The ratio of any two corresponding surfaces is equal to the square of the ratio of any two corresponding lines.

(*c*) The ratio of any two corresponding volumes is equal to the cube of the ratio of any two corresponding lines.

9. The fundamental law of machines stated in the form of the law of the lever, $WD = W_1D_1$.

10. The intensity of illumination (or of heat) received by any parallel surfaces varies inversely as the square of their distances from the source.

11. The weight of an object is inversely proportional to the square of its distance from the center of the earth.

† Among similar figures may be included triangles, polygons, circles, prisms, cylinders, pyramids, cones, and spheres; and also scale drawings, maps, and other plane and solid figures of irregular outline.

12. The strength of a beam of fixed length is given by the formula $S = kbd^2$ in which b is the breadth and d the depth.

13. The distance through which a body falls varies as the square of the time during which it falls. (The resistance of the air is disregarded.)

14. The rated horse power of a gasoline engine is given by the formula $H = \frac{2}{5} Nd^2$ in which N is the number of cylinders and d is the diameter or bore of each cylinder.

54, B. Problems

Solve and check the following problems. Refer if necessary to the formulas of Exercise A.

1. On a certain map the distance from Boston to New York is 4.1″, and the distance from New York to Chicago is 18″. On a blackboard copy of this map the first of these distances is 2′ 7″. How long should the second distance be? Answer to the nearest inch.

2. The sides of an angle of a triangle are 7 in. and 15 in. respectively. Find the lengths of the segments into which the bisector of this angle divides the opposite side which is 18 in. long. (The bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides.)

3. A triangle 12 in. high contains 256 sq. in. What will be the area of a similar triangle 15 in. high?

4. An object which weighs 150 lb. at the surface of the earth will weigh how much when 50 miles above the surface? Answer to nearest tenth of a pound. (Radius of earth 4000 mi.)

5. A stone pyramid 50 ft. high weighs 1000 tons. It is required to build a 2000-ton pyramid of the same shape. How high should it be made?

***6.** The top of a desk, 9 ft. below the ceiling, is insufficiently lighted by a lamp directly overhead and 1 ft. below the ceiling. If the lamp is lowered 3 ft., how will the lighting be affected? Answer by saying that it will be —— times as intense as before.

***7.** The total steam pressure on a piston 10 in. in diameter is 3300 lb. If the steam pressure is unchanged, what will be the total pressure on a piston 14 in. in diameter?

8. The safe load for a certain 10 ft. beam which is 2 in. wide and 6 in. deep is 1100 lb. What is the safe load for a 10 ft. beam of the same material 3 in. wide and 8 in. deep?

9. A bar 84 in. long carries a load of 110 lb. on one end and a load of 230 lb. on the other. At what point will the bar balance? Disregard the weight of the bar and answer to the nearest inch.

***10.** The time of vibration of a pendulum varies directly as the square root of the length. If the length is 99.3 cm. when the time is 1 second, how long should a pendulum be made in order to vibrate in 2 seconds? In $\frac{1}{2}$ second? (Use logarithms if convenient.)

***11.** Electric linemen in stretching a wire to the correct tension measure the tension by swinging the wire and observing the rate of vibration. One of the formulas used is

$$n = \frac{1}{2\,l}\sqrt{\frac{t}{w}}$$

in which l is the length, t the tension, and w the weight per unit length of the wire. What is the law of variation for n? Solve the formula for t. Assuming that l and w are constants, express the law of dependence of t on n in the k form. What is the value of k in terms of l and w?

***12.** For any two planets the squares of the times of rotation are proportional to the cubes of the distances from the sun.

Mercury rotates about the sun in 88 days and Neptune in 165 years. The earth is 92,500,000 miles from the sun. How far from the sun is Mercury? Neptune? In the computation use logarithms. Give the results to three-figure accuracy.

EXERCISES FOR CHAPTER IX

SYSTEMS OF LINEAR EQUATIONS

Exercise 55. Linear Systems in Two Unknowns

(See page 275.)

55, A. Literal Systems

Solve for x and y. Check by substituting the roots in the original equations or by numerical substitution.

1. $x + y = a \qquad x - y = b$

2. $x + 5y = 3c \qquad 2x - 3y = 4c$

3. $y = k_1x + b_1 \qquad y = k_2x + b_2$

4. $ax + by - 2 = 0 \qquad a^2x - b^2y = a - b$

5. $\dfrac{x}{y} = \dfrac{r}{s} \qquad \dfrac{x+1}{y+1} = \dfrac{t}{s}$

6. $ry - sx = r^2 + s^2 \qquad rx + by = r^2 + s^2$

***7.** $ax + by = 2 \qquad ab(ay - bx) - a^2 + b^2 = 0$

***8.** $x + y = \dfrac{2r}{r^2 - s^2} \qquad rx + sy = \dfrac{r^2 + s^2}{r^2 - s^2}$

***9.** $bx + cy = b^2 + 2bc - c^2 \qquad cx + by = c^2 + b^2$

***10.** $mx + ny = 2 \qquad \dfrac{x+y}{m+n} = \dfrac{1}{mn}$

55, B

Solve and check each determinate system. Eliminate by combination or by substitution or solve by a formula or by a graph.

1. $3x + 6y = 12 \qquad 2x + 4y = 8$
2. $5x + 10y = 20 \qquad 3x + 6y = 12$
3. $12x + 6y = 9 \qquad 6x + 12y = 18$
4. $14a + 22b = 17 \qquad 21a + 33b = 27$
5. $3a + b = 27 \qquad a + 3b = 9$
6. $34r - 4s = 32 \qquad 51r + 6s = 48$
7. $7a - 21a' = 42 \qquad 12a - 4a' = 24$
8. $4y - 2x = 12 \qquad 3y - 6x = 9$
9. $14x - 6y = 26 \qquad 9y - 21x = -39$
10. $12x - 96y = 148 \qquad 9x - 72y = 111$
11. $x - y - 2z = 15 \qquad 3x - 5y = 18$
12. $\frac{a}{8} = \frac{5}{3} + \frac{b}{6} \qquad \frac{a}{4} + \frac{b}{8} = \frac{3}{2}$
13. $\frac{c+d}{3} = 15 - d$

    $\frac{c+d}{5} + d = 13$
14. $\frac{4a + b - 16}{2a - 3b + 5} = 4$

    $\frac{15 + 12a - 7b}{5a + b - 9} = 6$
15. $3x + y = \frac{2x - 1}{2}$

    $2x + y = \frac{1}{2}$
16. $6x + 4y = 2$

    $.3x + .1y = .1$

*17. $\frac{x}{2} = 12 + \frac{y+32}{4}$

$\frac{y}{8} = 25 - \frac{3x-2y}{9}$

*18. $\frac{2x}{3} - \frac{3y}{4} = 10$

$0.6x = -0.2 - 0.7y$

*19. $\frac{2a-5}{4a+5} = \frac{3b+2}{6b-4}$

$5 = 3a + 10b$

*20. $3(3a+4) = 10a_1 - 15$

$\frac{6a+17}{5} - \frac{9a+1}{2} = -\frac{5a_1-6}{3}$

*21. $\frac{2(5x+3)}{12} - \frac{2x+y}{2} = \frac{2x-3y}{9}$

$\frac{2(3x+1)}{5} - \frac{4x-3y}{3} - \frac{5y-4}{4} = 0$

*22. $\frac{4a+3b}{2} - \frac{8a-9b}{8} - \frac{7}{4} = 0$

$6a - 5b + \frac{2a-7b}{3} = -\frac{14}{9}$

Exercise 56. Linear Systems in Three Unknowns

(See page 276.)

Solve and check. Work systematically.

1. $4a - b + c = 1 \qquad a + 2b + 7c = 7 \qquad 3a - b - 5c = 5$

2. $3x - 2y + z = 11 \qquad x - y - 2z + 3 = 0$
$4x + 2y + 3z = 15$

3. $2a - 3a_1 - a_2 = -7 \qquad 11a - 11a_1 - 2a_2 + 3 = 0$
$3a + 7a_1 + a_2 = 30$

4. $7a + 5a' - 3a'' = 26 \qquad 6a + 2a' - 5a'' = 13$

$3a + 3a' - 2a'' = 13$

5. $2y + x + z = 0 \qquad 3y - 2x + 3z = 4$

$5y + 7x + 2z = 2$

6. $x - 2y + 3z = 6 \qquad 2x + 3y - 4z = 20$

$3x - 2y + 5z = 26$

***7.** $\dfrac{2}{a-3} = \dfrac{1}{b-2} \qquad \dfrac{b-1}{3} = \dfrac{2b-c}{6} \qquad \dfrac{2+2c}{2a} = \dfrac{5}{6}$

***8.** $\dfrac{1}{x} + \dfrac{1}{y} = 1 \qquad \dfrac{1}{y} + \dfrac{1}{z} = -3 \qquad \dfrac{1}{z} + \dfrac{1}{x} = 2$

***9.** $\dfrac{3}{x} + \dfrac{6}{y} = 3 \qquad \dfrac{2}{x} + \dfrac{3}{z} = 2 \qquad \dfrac{9}{z} + \dfrac{4}{y} = 4$

***10.** $\dfrac{6}{r} - \dfrac{8}{s} - \dfrac{16}{t} = 10 \qquad \dfrac{4}{r} + \dfrac{3}{s} + \dfrac{9}{t} = 9 \qquad \dfrac{3}{r} - \dfrac{5}{s} - \dfrac{7}{t} = 5$

***11.** $\dfrac{ab}{a-b} = \dfrac{1}{8} \qquad \dfrac{bc}{b-c} = \dfrac{1}{4} \qquad \dfrac{ca}{c+a} = \dfrac{1}{16}$

***12.** $a + b = 15 + c \qquad a - b = 5 \qquad a = 7 - c$

***13.** $b + c = 5.9 - a \qquad a - c = 3.1 - b \qquad a - b = 3.3 - c$

***14.** $x + y + z = r + s + t \qquad x + 3y + 4z = r + 3s$

$x + 2y + 3z = r + 2s$

***15.** $x + y - z = 0 \qquad x - y = 2s \qquad x + z = 3r + s$

***16.** $\frac{5}{2}w - x - 2y = 9 \qquad 3w - 4y = 0$

$\dfrac{3w - 2y}{4} = x + 6\frac{1}{2}$

***17.** $w + x + y + z = 0 \qquad 2w - x + 4y - 3z = 20$

$-4w - 5x + y - z = 1 \qquad w + 3x - 7y + 2z = -24$

EXERCISES FOR CHAPTER X

QUADRATIC EQUATIONS

Exercise 57. Quadratic Equations

(See page 287, and observe particularly the cautions listed on 285.)

57, A. Solution. Special Cases

1. Solve the equation $6x^2 - 13x + 6 = 0$ in four ways.

***2.** Solve by completing the square: $ex^2 + fx + g = 0$.

Solve the following equations by appropriate methods. Extract irrational roots to three-figure accuracy. Give most attention to the methods on which you need most practice. Comment on the special cases.

3. $x - 4 = 0$ **4.** $x^2 - 4 = 0$ **5.** $x^2 - 11 = 0$

6. $x^2 - 18 = 0$ **7.** $(4 + x)(4 - x) = 9$

8. $18a^2 - (2a^2 + 3) = 6$ **9.** $7a^2 + 2a = 32$

10. $28 = \dfrac{x^2}{4} - \dfrac{2x}{3}$ **11.** $\dfrac{1}{4} = \dfrac{4}{x^2 - 2x + 1}$

12. $\dfrac{x^2}{x+3} + \dfrac{9}{x+3} = 8$ ***13.** $2x + 2 = \dfrac{3(x-1)}{x+3}$

***14.** $\frac{1}{3}x(x-1) - \frac{1}{4}(x^2 + x - 2) = x - 7$

15. $x - 5 = \dfrac{2(x+1)}{x-4}$

16. $\dfrac{x-5}{x^2-25} = \dfrac{1}{11}$ Solve by clearing of fractions in the first step. Also solve by reducing to lower terms in the first step. Which method introduces an extraneous root? How?

***17.** $\dfrac{a^3 - 8}{a^2 - 4} = a + 3$ ***18.** $1 - \dfrac{3}{a+3} = \dfrac{15}{(a-2)(a+3)}$

*19. $\dfrac{2}{b-2} - 1 = \dfrac{2}{(b-1)(b-2)}$

20. $3x - 1 = x + 3$

21. $\sqrt{x+1} = 5 - x$

22. $a + \sqrt{26 - a} = 6$

23. $7 + \sqrt{x^2 - 16} = 2x$

57, B. Equations in Quadratic Form

(Observe the cautions listed on page 285.)

1. $x^4 - 2x^2 - 35 = 0$

2. $x^4 - 6x^2 + 9 = 0$

3. $x^{\frac{2}{3}} - x^{\frac{1}{3}} - 6 = 0$

4. $x^6 + 8 = 9x^3$

5. $(x-3)^2 - (x-3) = 12$

6. $(x-3) - (x-3)^{\frac{1}{2}} = 12$

7. $x - 3 - \sqrt{x-3} - 20 = 0$

8. $x - 3\sqrt{x-3} - 13 = 0$

9. $x - 2\sqrt{x-3} - 18 = 0$

*10. $2x^2 - x + 1 + 2\sqrt{2x^2 - x + 1} = 8$

*11. $2x^2 - x + \sqrt{2x^2 - x + 1} - 5 = 0$

*12. $2x^2 + 16x - 3\sqrt{x^2 + 8x + 2} - 1 = 0$

57, C. Other Equations Containing Radicals

(Observe the cautions listed on page 285.)

Solve and check. (If you can rewrite any of these equations in quadratic form, do so. If not, arrange in suitable form, and then square.) Use principal square roots only in checking.

1. $x - 5\sqrt{x-1} = -5$

2. $\sqrt{5x + 11} - \sqrt{3x + 1} = 2$

3. $\sqrt{y - 3} + \sqrt{2y + 1} = \sqrt{5y - 4}$

*4. $\sqrt{m} + \sqrt{4 + 3m} = 6$

*5. $\sqrt{2c - d + 2x} - 4\sqrt{c - d} = \sqrt{10c - 9d - 6x}$

*6. $\sqrt{x - 3} + \sqrt{2x + 4} = \sqrt{3x + 1}$

Exercise 58. Relations between Constants and Roots in Quadratic Equations

58, A

(See page 290.)

Write quadratic equations of which the roots are given below:

1. $3, 4$
2. $3, -4$
3. $-3, -4$
4. $4, 4$
5. $4, -\frac{1}{2}$
6. $a, 3a$
7. $2a, -3a$
8. $a + b, a - b$
9. $3 \pm \sqrt{2}$
10. $1 \pm \sqrt{-5}$
11. $a \pm \sqrt{3b}$
12. $-1.9, 2.1$

13. One of the roots of $x^2 - \frac{11}{2}x + \frac{15}{2} = 0$ is 3. Find the other root in three ways.

14. One root of $3x^2 - 2x - 7 = 0$ is $\frac{1}{3} - \frac{1}{3}\sqrt{22}$. Find the other root in three ways.

15. Are $\frac{2}{3}$ and $-\frac{3}{2}$ roots of $2x^2 - 3x - 2 = 0$? Prove your answer by two different lines of reasoning.

16. For what value of k are the roots of $3x^2 + kx + 3k = 0$ real and equal? ($a = 3$, $b = k$, $c = 3k$. What is the discriminant?)

17. One root of the equation $x^2 - kx + 6 = 0$ is 3. Find k.

18. One root of the equation $x^2 + 3x + 2k = 0$ is 5. Find k.

19. One root of the equation $x^2 + kx + 40 = 0$ exceeds the other root by 3. Find k.

20. One root of the equation $x^2 - 15x + k = 0$ is twice the other. Find k.

21. Write the equation of which the roots are $a + b$ and $a - b$. Check by solving the equation by formula.

58, B. The Kinds of Numbers Involved in the Roots of Quadratic Equations

Without solving, give the character of the roots of each equation below:

1. $4x^2 + 12x + 7 = 0$

2. $4x^2 - 12x + 9 = 0$

3. $4x^2 - 12x + 10 = 0$

4. $4x^2 - 12x + 8 = 0$

5. Make a set of quadratic equations having roots of different character similar to those of examples 1–4.

***6.** In the equation $x^2 + (k + 1)x + k^2 = 0$, determine value or values for k which will give the equation (a) equal roots, (b) one root equal to zero.

Exercise 59. Systems of Equations Involving Quadratics

A. Linear-Quadratic Pairs

(See page 291.)

Solve and check each pair of equations below. Observe the cautions mentioned or referred to in example 1, page 290.

1. $x^2 - 2xy + y^2 = 4$
$x + y = 8$

2. $x^2 - xy + y^2 = 7$
$x + y = 4$

3. $x - y = -1$
$3x^2 + xy + 3y^2 = 17$

4. $4xy - x^2 = 27$
$x + 2y = 15$

5. $a^2 - ab + 3b^2 = 36$
$2a + 3b = 18$

6. $2a^2 + b^2 = 34$
$2a - b = 2$

7. $a - b = 1$
$a^2 + 9b^2 = 45$

8. $a^2 + b^2 + 4a = 9$
$2a + b = 4$

9. $b^2 - 11 = ab + a^2$
$2b - 3a = 4$

10. $3x - y = 3$
$7x^2 - y^2 = 27$

***11.** $8b = 5 - 5a$
$2b - 5 = 3b^2 + 2ab$

***12.** $2x + 3y = 14$
$x^2 = y^2 + 12$

***13.** $5a - 3b + 1 = 0$
$2b^2 + 3ab - 5a^2 + 7a - 6ab = 4$

14. $\frac{x}{5} + \frac{y}{3} = 1$
$\frac{x^2}{25} + \frac{y^2}{9} = 1$

***15.** $\frac{2}{x} + \frac{1}{y} = 0$
$\frac{x+3}{y} + 2 - \frac{2}{x} = \frac{2}{xy}$

16. From the formulas $v = gt$ and $S = \frac{1}{2}gt^2$, eliminate t.

17. In a cone the volume is one third the base times the height, and the base is π times the square of the radius. Express these two facts as formulas and deduce a formula for the volume in terms of the radius and the height.

59, B. The Quadratic Pair. Special Cases

(See page 292.)

Solve and check:

***1.** $a^2 + b^2 = 40$
$ab = 12$

***2.** $xy = 2$
$4x^2 + y^2 = 17$

***3.** $4a^2 + 9b^2 = 2$
$6ab - 1 = 0$

***4.** $b^2 = 5 + 2ab$
$a^2 + b^2 = 29$

***5.** $x^2 = (5 - y)(5 + y)$
$xy = 12$

***6.** $\frac{a^2}{4} + \frac{b^2}{9} = 1$
$ab = \frac{4\sqrt{2}}{3}$

***7.** $3a^2 + 2b^2 = 44$
$a^2 = b$

***8.** $x^2 - y^2 = -12$
$8x = y^2$

*9. $m^2 + n^2 = 25$
$mn + n^2 = 21$

*10. $x^2 - 2y^2 = 24$
$2x^2 - 5y^2 = 148$

*11. $4a^2 + 3b^2 = 628$
$6a^2 - 2b^2 = 6$

*12. $2f^2 - 4g^2 = .32$
$5f^2 - 7g^2 = 2.72$

*13. $a^2 + b^2 = 50$
$a^2 + ab + b^2 = 57$

*14. $x^2 - xy + y^2 = 21$
$x^2 + y^2 = 26$

*15. $\frac{3}{a^2} + \frac{2}{b^2} = \frac{5}{6}$
$\frac{9}{a^2} + \frac{4}{b^2} = 2$

*16. $x^2 + y^2 = 13$
$\frac{1}{xy} = -\frac{1}{6}$

Exercise 60. Problems

(See page 294.)

*1. Three times the square of a number is equal to 15 times the number itself. What is the number?

*2. On a stream flowing 2 miles an hour, a man takes 8 hours longer to row 24 miles upstream than to row the same distance down. How fast can he row in still water?

*3. A rectangular swimming pool is surrounded by a walk 5 ft. wide. The area of the pool is 2400 sq. ft., and that of the walk half as much. Find the dimensions of the pool.

*4. Derive a formula for the area of any equilateral triangle a side of which is x inches.

*5. A circular flower bed is surrounded by a walk 7 ft. wide and the area of the walk is $\frac{7}{9}$ the area of the bed. Find the diameter of the flower bed. (Take $\pi = \frac{22}{7}$.)

*6. It is desired to divide a line 20 in. long into segments so that the longer shall be the mean proportional between the shorter and the whole line. What will be the length of each segment?

***7.** The hypotenuse of a right triangle is 10 in., and the sum of the other two sides 14 in. Find these two sides.

***8.** The combined perimeter of two squares is 48 in., and one contains 24 sq. ft. more than the other. How large are they?

***9.** The product of a certain two-digit number and the sum of its digits is 729. If 63 be subtracted from the number, its digits will be reversed. What is the number?

***10.** Eight boys, seniors and freshmen, paid $30 for a radio set. The seniors paid half and the freshmen half, but each senior paid $2 more than each freshman. How much did each pay?

***11.** The product of two numbers is 31 greater than their sum, and the sum of their squares is 106. What are the numbers?

***12.** A rectangular flower bed containing 54 sq. ft. was just doubled in area by the addition of a strip $1\frac{1}{2}$ ft. wide completely surrounding it. How large was the bed at first?

EXERCISES FOR CHAPTER XI

THE BINOMIAL FORMULA

Exercise 61. The Binomial Formula

61, A. Binomial Expansion. (See page 299.)

Expand:

1. $(r-s)^5$	**2.** $(c+d)^6$	**3.** $(x-y)^3$
4. $(3-a)^4$	**5.** $(5+x)^3$	**6.** $(x-3)^4$
7. $(a-2b)^5$	**8.** $(2a-b)^4$	**9.** $(x-\frac{1}{2}y)^6$
10. $(2c+3d)^4$	**11.** $2(c-3d)^4$	**12.** $(3x-y^2)^5$

Write the first three and the last three terms of:

13. $(x+y)^{40}$	**14.** $(x-y)^{50}$	**15.** $(x-7)^{15}$

(In example 15 do not expand the powers of 7. If needed, they could be found approximately by the use of logarithms.)

Expand:

16. $\left(\frac{a}{b} - \frac{b}{a}\right)^5$ **17.** $(2x + \frac{1}{2})^5$ **18.** $\left(\frac{1}{2a} - 2a\right)^5$

19. $(1.05)^4$ **20.** $(1.02)^5$ **21.** $[(a+b) - c]^3$

61, B. Finding Any Required Term

(See page 300.)

Find the term indicated:

1. The fifth term of $(x + y)^{10}$

2. The sixth term of $(r - s)^{14}$

3. The eighth term of $(h - k)^{18}$

4. The thirteenth term of $(s - t)^{23}$

5. The ninth term of $\left(a - \frac{1}{a}\right)^{28}$

6. The fifth term of $(a - 2b^2)^{11}$

7. The middle term of $\left(\frac{a}{2} - \frac{2}{a}\right)^{20}$

8. The two middle terms of $\left(\frac{s}{y} + \frac{y}{s}\right)^{15}$

9. The seventh term of $\left(\frac{2a}{3} - \frac{3}{2z}\right)^{12}$

***10.** The middle term of $\left(2a^5 + \frac{b^{\frac{1}{4}}}{3}\right)^{12}$

***11.** That term of $\left(\frac{3a}{4} - \frac{2}{3a}\right)^9$ which contains a^3 in the denominator.

*12. The term containing x^0 of $\left(\dfrac{x-2}{x}\right)^{12}$

*13. The term containing y^4 of $\left(\dfrac{3}{y}+y^2\right)^5$

61, C. Negative and Fractional Exponents

(See page 301.)

Expand as indicated:

*1. The first five terms of $(a+b)^{-1}$

*2. The first four terms of $(a-b)^{-2}$

*3. The first four terms of $(a+b)^{-3}$

*4. The first four terms of $(a^2-\frac{1}{2})^{-5}$

*5. The first four terms $(a^{-1}+b^{-1})^{-4}$

*6. $\sqrt{1-x} \equiv (1-x)^{\frac{1}{2}}$ to four terms.

*7. Find $\sqrt[4]{1.005}$ to three-figure accuracy.

*8. Find $\sqrt[3]{12} \equiv \sqrt[3]{8(1+\frac{1}{2})}$ to three-figure accuracy. Verify by the use of logarithms.

*9. Expand $(1-x)^{-1}$ to four terms. In the resulting series substitute $\frac{1}{2}$ for x. Notice that

$$(1-x)^{-1} \equiv \frac{1}{1-x}$$

In this fraction substitute $\frac{1}{2}$ for x and compare the result with that of the first substitution. Verify the result by the theory of geometric progression.

*10. In the preceding exercise, substitute 2 instead of $\frac{1}{2}$ and tell why the theory of geometric progression does not serve in the same way.

61, D. Using the Binomial Formula

(See page 302.)

1. Find the amount of \$330 at 4% compounded quarterly for 11 years. Verify so far as the table of logarithms will permit.

2. Find the amount of \$300 at 5% compounded annually for 6 years.

3. How much will \$500 in a savings bank at 4% compound interest (semi-annually) amount to in $3\frac{1}{2}$ years?

4. Find the interest on a loan of \$25,000 at 3% compounded annually if payment is made at the end of 5 years.

5. $200(1.02)^4$

6. $340(1.01)^6$

PART III

TABLES AND LOGARITHMS

[1] From *Four Place Tables*, by E. V. Huntington, Houghton Mifflin Company Used by permission of the author and publishers.

N.	0	1	2	3	4	5	6	7	8	9
1.0	1.000	1.020	1.040	1.061	1.082	1.102	1.124	1.145	1.166	1.188
1.1	1.210	1.232	1.254	1.277	1.300	1.322	1.346	1.369	1.392	1.416
1.2	1.440	1.464	1.488	1.513	1.538	1.562	1.588	1.613	1.638	1.664
1.3	1.690	1.716	1.742	1.769	1.796	1.822	1.850	1.877	1.904	1.932
1.4	1.960	1.988	2.016	2.045	2.074	2.102	2.132	2.161	2.190	2.220
1.5	2.250	2.280	2.310	2.341	2.372	2.402	2.434	2.465	2.496	2.528
1.6	2.560	2.592	2.624	2.657	2.690	2.722	2.756	2.789	2.822	2.856
1.7	2.890	2.924	2.958	2.993	3.028	3.062	3.098	3.133	3.168	3.204
1.8	3.240	3.276	3.312	3.349	3.386	3.422	3.460	3.497	3.534	3.572
1.9	3.610	3.648	3.686	3.725	3.764	3.802	3.842	3.881	3.920	3.960
2.0	4.000	4.040	4.080	4.121	4.162	4.202	4.244	4.285	4.326	4.368
2.1	4.410	4.452	4.494	4.537	4.580	4.622	4.666	4.709	4.752	4.796
2.2	4.840	4.884	4.928	4.973	5.018	5.062	5.108	5.153	5.198	5.244
2.3	5.290	5.336	5.382	5.429	5.476	5.522	5.570	5.617	5.664	5.712
2.4	5.760	5.808	5.856	5.905	5.954	6.002	6.052	6.101	6.150	6.200
2.5	6.250	6.300	6.350	6.401	6.452	6.502	6.554	6.605	6.656	6.708
2.6	6.760	6.812	6.864	6.917	6.970	7.022	7.076	7.129	7.182	7.236
2.7	7.290	7.344	7.398	7.453	7.508	7.562	7.618	7.673	7.728	7.784
2.8	7.840	7.896	7.952	8.009	8.066	8.122	8.180	8.237	8.294	8.352
2.9	8.410	8.468	8.526	8.585	8.644	8.702	8.762	8.821	8.880	8.940
3.0	9.000	9.060	9.120	9.181	9.242	9.302	9.364	9.425	9.486	9.548
3.1	9.610	9.672	9.734	9.797	9.860	9.922	9.986	10.05	10.11	10.18
3.2	10.24	10.30	10.37	10.43	10.50	10.56	10.63	10.69	10.76	10.82
3.3	10.89	10.96	11.02	11.09	11.16	11.22	11.29	11.36	11.42	11.49
3.4	11.56	11.63	11.70	11.76	11.83	11.90	11.97	12.04	12.11	12.18
3.5	12.25	12.32	12.39	12.46	12.53	12.60	12.67	12.74	12.82	12.89
3.6	12.96	13.03	13.10	13.18	13.25	13.32	13.40	13.47	13.54	13.62
3.7	13.69	13.76	13.84	13.91	13.99	14.06	14.14	14.21	14.29	14.36
3.8	14.44	14.52	14.59	14.67	14.75	14.82	14.90	14.98	15.05	15.13
3.9	15.21	15.29	15.37	15.44	15.52	15.60	15.68	15.76	15.84	15.92
4.0	16.00	16.08	16.16	16.24	16.32	16.40	16.48	16.56	16.65	16.73
4.1	16.81	16.89	16.97	17.06	17.14	17.22	17.31	17.39	17.47	17.56
4.2	17.64	17.72	17.81	17.89	17.98	18.06	18.15	18.23	18.32	18.40
4.3	18.49	18.58	18.66	18.75	18.84	18.92	19.01	19.10	19.18	19.27
4.4	19.36	19.45	19.54	19.62	19.71	19.80	19.89	19.98	20.07	20.16
4.5	20.25	20.34	20.43	20.52	20.61	20.70	20.79	20.88	20.98	21.07
4.6	21.16	21.25	21.34	21.44	21.53	21.62	21.72	21.81	21.90	22.00
4.7	22.09	22.18	22.28	22.37	22.47	22.56	22.66	22.75	22.85	22.94
4.8	23.04	23.14	23.23	23.33	23.43	23.52	23.62	23.72	23.81	23.91
4.9	24.01	24.11	24.21	24.30	24.40	24.50	24.60	24.70	24.80	24.90
5.0	25.00	25.10	25.20	25.30	25.40	25.50	25.60	25.70	25.81	25.91
5.1	26.01	25.11	26.21	26.32	26.42	26.52	26.63	26.73	26.83	26.94
5.2	27.04	27.14	27.25	27.35	27.46	27.56	27.67	27.77	27.88	27.98
5.3	28.09	28.20	28.30	28.41	28.52	28.62	28.73	28.84	28.94	29.05
5.4	29.16	29.27	29.38	29.48	29.59	29.70	29.81	29.92	30.03	30.14

N.	0	1	2	3	4	5	6	7	8	9
5.5	30.25	30.36	30.47	30.58	30.69	30.80	30.91	31.02	31.14	31.25
5.6	31.36	31.47	31.58	31.70	31.81	31.92	32.04	32.15	32.26	32.38
5.7	32.49	32.60	32.72	32.83	32.95	33.06	33.18	33.29	33.41	33.52
5.8	33.64	33.76	33.87	33.99	34.11	34.22	34.34	34.46	34.57	34.69
5.9	34.81	34.93	35.05	35.16	35.28	35.40	35.52	35.64	35.76	35.88
6.0	36.00	36.12	36.24	36.36	36.48	36.60	36.72	36.84	36.97	37.09
6.1	37.21	37.33	37.45	37.58	37.70	37.82	37.95	38.07	38.19	38.32
6.2	38.44	38.56	38.69	38.81	38.94	39.06	39.19	39.31	39.44	39.56
6.3	39.69	39.82	39.94	40.07	40.20	40.32	40.45	40.58	40.70	40.83
6.4	40.96	41.09	41.22	41.34	41.47	41.60	41.73	41.86	41.99	42.12
6.5	42.25	42.38	42.51	42.64	42.77	42.90	43.03	43.16	43.30	43.43
6.6	43.56	43.69	43.82	43.96	44.09	44.22	44.36	44.49	44.62	44.76
6.7	44.89	45.02	45.16	45.29	45.43	45.56	45.70	45.83	45.97	46.10
6.8	46.24	46.38	46.51	46.65	46.79	46.92	47.06	47.20	47.33	47.47
6.9	47.61	47.75	47.89	48.02	48.16	48.30	48.44	48.58	48.72	48.86
7.0	49.00	49.14	49.28	49.42	49.56	49.70	49.84	49.98	50.13	50.27
7.1	50.41	50.55	50.69	50.84	50.98	51.12	51.27	51.41	51.55	51.70
7.2	51.84	51.98	52.13	52.27	52.42	52.56	52.71	52.85	53.00	53.14
7.3	53.29	53.44	53.58	53.73	53.88	54.02	54.17	54.32	54.46	54.61
7.4	54.76	54.91	55.06	55.20	55.35	55.50	55.65	55.80	55.95	56.10
7.5	56.25	56.40	56.55	56.70	56.85	57.00	57.15	57.30	57.46	57.61
7.6	57.76	57.91	58.06	58.22	58.37	58.52	58.68	58.83	58.98	59.14
7.7	59.29	59.44	59.60	59.75	59.91	60.06	60.22	60.37	60.53	60.68
7.8	60.84	61.00	61.15	61.31	61.47	61.62	61.78	61.94	62.09	62.25
7.9	62.41	62.57	62.73	62.88	63.04	63.20	63.36	63.52	63.68	63.84
8.0	64.00	64.16	64.32	64.48	64.64	64.80	64.96	65.12	65.29	65.45
8.1	65.61	65.77	65.93	66.10	66.26	66.42	66.59	66.75	66.91	67.08
8.2	67.24	67.40	67.57	67.73	67.90	68.06	68.23	68.39	68.56	68.72
8.3	68.89	69.06	69.22	69.39	69.56	69.72	69.89	70.06	70.22	70.39
8.4	70.56	70.73	70.90	71.06	71.23	71.40	71.57	71.74	71.91	72.08
8.5	72.25	72.42	72.59	72.76	72.93	73.10	73.27	73.44	73.62	73.79
8.6	73.96	74.13	74.30	74.48	74.65	74.82	75.00	75.17	75.34	75.52
8.7	75.69	75.86	76.04	76.21	76.39	76.56	76.74	76.91	77.09	77.26
8.8	77.44	77.62	77.79	77.97	78.15	78.32	78.50	78.68	78.85	79.03
8.9	79.21	79.39	79.57	79.74	79.92	80.10	80.28	80.46	80.64	80.82
9.0	81.00	81.18	81.36	81.54	81.72	81.90	82.08	82.26	82.45	82.63
9.1	82.81	82.99	83.17	83.36	83.54	83.72	83.91	84.09	84.27	84.46
9.2	84.64	84.82	85.01	85.19	85.38	85.56	85.75	85.93	86.12	86.30
9.3	86.49	86.68	86.86	87.05	87.24	87.42	87.61	87.80	87.98	88.17
9.4	88.36	88.55	88.74	88.92	89.11	89.30	89.49	89.68	89.87	90.06
9.5	90.25	90.44	90.63	90.82	91.01	91.20	91.39	91.58	91.78	91.97
9.6	92.16	92.35	92.54	92.74	92.93	93.12	93.32	93.51	93.70	93.90
9.7	94.09	94.28	94.48	94.67	94.87	95.06	95.26	95.45	95.65	95.84
9 8	96.04	96.24	96.43	96.63	96.83	97.02	97.22	97.42	97.61	97.81
9.9	98.01	98.21	98.41	98.60	98.80	99.00	99.20	99.40	99.60	99.80

N.	0	1	2	3	4	5	6	7	8	9
1.0	1.000	1.030	1.061	1.093	1.125	1.158	1.191	1.225	1.260	1.295
1.1	1.331	1.368	1.405	1.443	1.482	1.521	1.561	1.602	1.643	1.685
1.2	1.728	1.772	1.816	1.861	1.907	1.953	2.000	2.048	2.097	2.147
1.3	2.197	2.248	2.300	2.353	2.406	2.460	2.515	2.571	2.628	2.686
1.4	2.744	2.803	2.863	2.924	2.986	3.049	3.112	3.177	3.242	3.308
1.5	3.375	3.443	3.512	3.582	3.652	3.724	3.796	3.870	3.944	4.020
1.6	4.096	4.173	4.252	4.331	4.411	4.492	4.574	4.657	4.742	4.827
1.7	4.913	5.000	5.088	5.178	5.268	5.359	5.452	5.545	5.640	5.735
1.8	5.832	5.930	6.029	6.128	6.230	6.332	6.435	6.539	6.645	6.751
1.9	6.859	6.968	7.078	7.189	7.301	7.415	7.530	7.645	7.762	7.881
2.0	8.000	8.121	8.242	8.365	8.490	8.615	8.742	8.870	8.999	9.129
2.1	9.261	9.394	9.528	9.664	9.800	9.938	10.078	10.22	10.36	10.50
2.2	10.65	10.79	10.94	11.09	11.24	11.39	11.54	11.70	11.85	12.01
2.3	12.17	12.33	12.49	12.65	12.81	12.98	13.14	13.31	13.48	13.65
2.4	13.82	14.00	14.17	14.35	14.53	14.71	14.89	15.07	15.25	15.44
2.5	15.63	15.81	16.00	16.19	16.39	16.58	16.78	16.97	17.17	17.37
2.6	17.58	17.78	17.98	18.19	18.40	18.61	18.82	19.03	19.25	19.47
2.7	19.68	19.90	20.12	20.35	20.57	20.80	21.02	21.25	21.48	21.72
2.8	21.95	22.19	22.43	22.67	22.91	23.15	23.39	23.64	23.89	24.14
2.9	24.39	24.64	24.90	25.15	25.41	25.67	25.93	26.20	26.46	26.73
3.0	27.00	27.27	27.54	27.82	28.09	28.37	28.65	28.93	29.22	29.50
3.1	29.79	30.08	30.37	30.66	30.96	31.26	31.55	31.86	32.16	32.46
3.2	32.77	33.08	33.39	33.70	34.01	34.33	34.65	34.97	35.29	35.61
3.3	35.94	36.26	36.59	36.93	37.26	37.60	37.93	38.27	38.61	38.96
3.4	39.30	39.65	40.00	40.35	40.71	41.06	41.42	41.78	42.14	42.51
3.5	42.88	43.24	43.61	43.99	44.36	44.74	45.12	45.50	45.88	46.27
3.6	46.66	47.05	47.44	47.83	48.23	48.63	49.03	49.43	49.84	50.24
3.7	50.65	51.06	51.48	51.90	52.31	52.73	53.16	53.58	54.01	54.44
3.8	54.87	55.31	55.74	56.18	56.62	57.07	57.51	57.96	58.41	58.86
3.9	59.32	59.78	60.24	60.70	61.16	61.63	62.10	62.57	63.04	63.52
4.0	64.00	64.48	64.96	65.45	65.94	66.43	66.92	67.42	67.92	68.42
4.1	68.92	69.43	69.93	70.44	70.96	71.47	71.99	72.51	73.03	73.56
4.2	74.09	74.62	75.15	75.69	76.23	76.77	77.31	77.85	78.40	78.95
4.3	79.51	80.06	80.62	81.18	81.75	82.31	82.88	83.45	84.03	84.60
4.4	85.18	85.77	86.35	86.94	87.53	88.12	88.72	89.31	89.92	90.52
4.5	91.13	91.73	92.35	92.96	93.58	94.20	94.82	95.44	96.07	96.70
4.6	97.34	97.97	98.61	99.25	99.90	100.54	101.2	101.8	102.5	103.2
4.7	103.8	104.5	105.2	105.8	106.5	107.2	107.9	108.5	109.2	109.9
4.8	110.6	111.3	112.0	112.7	113.4	114.1	114.8	115.5	116.2	116.9
4.9	117.6	118.4	119.1	119.8	120.6	121.3	122.0	122.8	123.5	124.3
5.0	125.0	125.8	126.5	127.3	128.0	128.8	129.6	130.3	131.1	131.9
5.1	132.7	133.4	134.2	135.0	135.8	136.6	137.4	138.2	139.0	139.8
5.2	140.6	141.4	142.2	143.1	143.9	144.7	145.5	146.4	147.2	148.0
5.3	148.9	149.7	150.6	151.4	152.3	153.1	154.0	154.9	155.7	156.6
5.4	157.5	158.3	159.2	160.1	161.0	161.9	162.8	163.7	164.6	165.5

N.	0	1	2	3	4	5	6	7	8	9
5.5	166.4	167.3	168.2	169.1	170.0	171.0	171.9	172.8	173.7	174.7
5.6	175.6	176.6	177.5	178.5	179.4	180.4	181.3	182.3	183.3	184.2
5.7	185.2	186.2	187.1	188.1	189.1	190.1	191.1	192.1	193.1	194.1
5.8	195.1	196.1	197.1	198.2	199.2	200.2	201.2	202.3	203.3	204.3
5.9	205.4	206.4	207.5	208.5	209.6	210.6	211.7	212.8	213.8	214.9
6.0	216.0	217.1	218.2	219.3	220.3	221.4	222.5	223.6	224.8	225.9
6.1	227.0	228.1	229.2	230.3	231.5	232.6	233.7	234.9	236.0	237.2
6.2	238.3	239.5	240.6	241.8	243.0	244.1	245.3	246.5	247.7	248.9
6.3	250.0	251.2	252.4	253.6	254.8	256.0	257.3	258.5	259.7	260.9
6.4	262.1	263.4	264.6	265.8	267.1	268.3	269.6	270.8	272.1	273.4
6.5	274.6	275.9	277.2	278.4	279.7	281.0	282.3	283.6	284.9	286.1
6.6	287.5	288.8	290.1	291.4	292.8	294.1	295.4	296.7	298.1	299.2
6.7	300.8	302.1	303.5	304.8	306.2	307.5	308.9	310.3	311.7	313.0
6.8	314.4	315.8	317.2	318.6	320.0	321.4	322.8	324.2	325.7	327.4
6.9	328.5	329.9	331.4	332.8	334.3	335.7	337.2	338.6	340.1	341.5
7.0	343.0	344.5	345.9	347.4	348.9	350.4	351.9	353.4	354.9	356.4
7.1	357.9	359.4	360.9	362.5	364.0	365.5	367.1	368.6	370.1	371.7
7.2	373.2	374.8	376.4	377.9	379.5	381.1	382.7	384.2	385.8	387.4
7.3	389.0	390.6	392.2	393.8	395.4	397.1	398.7	400.3	401.9	403.6
7.4	405.2	406.9	408.5	410.2	411.8	413.5	415.2	416.8	418.5	420.2
7.5	421.9	423.6	425.3	427.0	428.7	430.4	432.1	433.8	435.5	437.2
7.6	439.0	440.7	442.5	444.2	445.9	447.7	449.5	451.2	453.0	454.8
7.7	456.5	458.3	460.1	461.9	463.7	465.5	467.3	469.1	470.9	472.7
7.8	474.6	476.4	478.2	480.0	481.9	483.7	485.6	487.4	489.3	491.2
7.9	493.0	494.9	496.8	498.7	500.6	502.5	504.4	506.3	508.2	510.1
8.0	512.0	513.9	515.8	517.8	519.7	521.7	523.6	525.6	527.5	529.5
8.1	531.4	533.4	535.4	537.4	539.4	541.3	543.3	545.3	547.3	549.4
8.2	551.4	553.4	555.4	557.4	559.5	561.5	563.6	565.6	567.7	569.7
8.3	571.8	573.9	575.9	578.0	580.1	582.2	584.3	586.4	588.5	590.6
8.4	592.7	594.8	596.9	599.1	601.2	603.4	605.5	607.6	609.8	612.0
8.5	614.1	616.3	618.5	620.7	622.8	625.0	627.2	629.4	631.6	633.8
8.6	636.1	638.3	640.5	642.7	645.0	647.2	649.5	651.7	654.0	656.2
8.7	658.5	660.8	663.1	665.3	667.6	669.9	672.2	674.5	676.8	679.2
8.8	681.5	683.8	686.1	688.5	690.8	693.2	695.5	697.9	700.2	702.6
8.9	705.0	707.3	709.7	712.1	714.5	716.9	719.3	721.7	724.2	726.6
9.0	729.0	731.4	733.9	736.3	738.8	741.2	743.7	746.1	748.6	751.1
9.1	753.6	756.1	758.6	761.0	763.6	766.1	768.6	771.1	773.6	776.2
9.2	778.7	781.2	783.8	786.3	788.9	791.5	794.0	796.6	799.2	801.8
9.3	804.4	807.0	809.6	812.2	814.8	817.4	820.0	822.7	825.3	827.9
9.4	830.6	833.2	835.9	838.6	841.2	843.9	846.6	849.3	852.0	854.7
9.5	857.4	860.1	862.8	865.5	868.3	871.0	873.7	876.5	879.2	882.0
9.6	884.7	887.5	890.3	893.1	895.8	898.6	901.4	904.2	907.0	909.9
9.7	912.7	915.5	918.3	921.2	924.0	926.9	929.7	932.6	935.4	938.3
9.8	941.2	944.1	947.0	949.9	952.8	955.7	958.6	961.5	964.4	967.4
9.9	970.3	973.2	976.2	979.1	982.1	985.1	988.0	991.0	994.0	997.0

Log

	0	1	2	3	4	5	6	7	8	9	10
1.0	0.0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	0414
1.1	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	0792
1.2	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	1139
1.3	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	1461
1.4	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	1761
1.5	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	2041
1.6	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	2304
1.7	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2553
1.8	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2788
1.9	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	3010
2.0	0.3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	3222
2.1	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	3424
2.2	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	3617
2.3	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	3802
2.4	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	3979
2.5	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	4150
2.6	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	4314
2.7	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	4472
2.8	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	4624
2.9	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	4771
3.0	0.4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	4914
3.1	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	5051
3.2	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	5185
3.3	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	5315
3.4	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	5441
3.5	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	5563
3.6	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	5682
3.7	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	5798
3.8	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	5911
3.9	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	6021
4.0	0.6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	6128
4.1	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	6232
4.2	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	6335
4.3	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	6435
4.4	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	6532
4.5	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	6628
4.6	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	6721
4.7	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	6812
4.8	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	6902
4.9	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	6990

Log

	0	1	2	3	4	5	6	7	8	9	10
5.0	0.6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	7076
5.1	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	7160
5.2	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	7243
5.3	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	7324
5.4	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	7404
5.5	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	7482
5.6	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	7559
5.7	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	7634
5.8	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	7709
5.9	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	7782
6.0	0.7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7853
6.1	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	7924
6.2	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	7993
6.3	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	8062
6.4	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	8129
6.5	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	8195
6.6	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	8261
6.7	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	8325
6.8	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	8388
6.9	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	8451
7.0	0.8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	8513
7.1	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	8573
7.2	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	8633
7.3	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	8692
7.4	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	8751
7.5	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	8808
7.6	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	8865
7.7	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	8921
7.8	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	8976
7.9	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	9031
8.0	0.9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	9085
8.1	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	9138
8.2	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	9191
8.3	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	9243
8.4	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	9294
8.5	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	9345
8.6	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	9395
8.7	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	9445
8.8	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	9494
8.9	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	9542
9.0	0.9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	9590
9.1	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	9638
9.2	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	9685
9.3	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	9731
9.4	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	9777
9.5	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	9823
9.6	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	9868
9.7	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	9912
9.8	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	9956
9.9	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	

Deg.	Sine 0′	10′	20′	30′	40′	50′	60′	
0	0.0000	0.0029	0.0058	0.0087	0.0116	0.0145	0.0175	89
1	0.0175	0.0204	0.0233	0.0262	0.0291	0.0320	0.0349	88
2	0.0349	0.0378	0.0407	0.0436	0.0465	0.0494	0.0523	87
3	0.0523	0.0552	0.0581	0.0610	0.0640	0.0669	0.0698	86
4	0.0698	0.0727	0.0756	0.0785	0.0814	0.0843	0.0872	85
5	0.0872	0.0901	0.0929	0.0958	0.0987	0.1016	0.1045	84
6	0.1045	0.1074	0.1103	0.1132	0.1161	0.1190	0.1219	83
7	0.1219	0.1248	0.1276	0.1305	0.1334	0.1363	0.1392	82
8	0.1392	0.1421	0.1449	0.1478	0.1507	0.1536	0.1564	81
9	0.1564	0.1593	0.1622	0.1650	0.1679	0.1708	0.1736	80
10	0.1736	0.1765	0.1794	0.1822	0.1851	0.1880	0.1908	79
11	0.1908	0.1937	0.1965	0.1994	0.2022	0.2051	0.2079	78
12	0.2079	0.2108	0.2136	0.2164	0.2193	0.2221	0.2250	77
13	0.2250	0.2278	0.2306	0.2334	0.2363	0.2391	0.2419	76
14	0.2419	0.2447	0.2476	0.2504	0.2532	0.2560	0.2588	75
15	0.2588	0.2616	0.2644	0.2672	0.2700	0.2728	0.2756	74
16	0.2756	0.2784	0.2812	0.2840	0.2868	0.2896	0.2924	73
17	0.2924	0.2952	0.2979	0.3007	0.3035	0.3062	0.3090	72
18	0.3090	0.3118	0.3145	0.3173	0.3201	0.3228	0.3256	71
19	0.3256	0.3283	0.3311	0.3338	0.3365	0.3393	0.3420	70
20	0.3420	0.3448	0.3475	0.3502	0.3529	0.3557	0.3584	69
21	0.3584	0.3611	0.3638	0.3665	0.3692	0.3719	0.3746	68
22	0.3746	0.3773	0.3800	0.3827	0.3854	0.3881	0.3907	67
23	0.3907	0.3934	0.3961	0.3987	0.4014	0.4041	0.4067	66
24	0.4067	0.4094	0.4120	0.4147	0.4173	0.4200	0.4226	65
25	0.4226	0.4253	0.4279	0.4305	0.4331	0.4358	0.4384	64
26	0.4384	0.4410	0.4436	0.4462	0.4488	0.4514	0.4540	63
27	0.4540	0.4566	0.4592	0.4617	0.4643	0.4669	0.4695	62
28	0.4695	0.4720	0.4746	0.4772	0.4797	0.4823	0.4848	61
29	0.4848	0.4874	0.4899	0.4924	0.4950	0.4975	0.5000	60
30	0.5000	0.5025	0.5050	0.5075	0.5100	0.5125	0.5150	59
31	0.5150	0.5175	0.5200	0.5225	0.5250	0.5275	0.5299	58
32	0.5299	0.5324	0.5348	0.5373	0.5398	0.5422	0.5446	57
33	0.5446	0.5471	0.5495	0.5519	0.5544	0.5568	0.5592	56
34	0.5592	0.5616	0.5640	0.5664	0.5688	0.5712	0.5736	55
35	0.5736	0.5760	0.5783	0.5807	0.5831	0.5854	0.5878	54
36	0.5878	0.5901	0.5925	0.5948	0.5972	0.5995	0.6018	53
37	0.6018	0.6041	0.6065	0.6088	0.6111	0.6134	0.6157	52
38	0.6157	0.6180	0.6202	0.6225	0.6248	0.6271	0.6293	51
39	0.6293	0.6316	0.6338	0.6361	0.6383	0.6406	0.6428	50
40	0.6428	0.6450	0.6472	0.6494	0.6517	0.6539	0.6561	49
41	0.6561	0.6583	0.6604	0.6626	0.6648	0.6670	0.6691	48
42	0.6691	0.6713	0.6734	0.6756	0.6777	0.6799	0.6820	47
43	0.6820	0.6841	0.6862	0.6884	0.6905	0.6926	0.6947	46
44	0.6947	0.6967	0.6988	0.7009	0.7030	0.7050	0.7071	45
	60′	50′	40′	30′	20′	10′	0′ Cosine	Deg.

Deg.	Sine 0′	10′	20′	30′	40′	50′	60′	
45	0.7071	0.7092	0.7112	0.7133	0.7153	0.7173	0.7193	44
46	0.7193	0.7214	0.7234	0.7254	0.7274	0.7294	0.7314	43
47	0.7314	0.7333	0.7353	0.7373	0.7392	0.7412	0.7431	42
48	0.7431	0.7451	0.7470	0.7490	0.7509	0.7528	0.7547	41
49	0.7547	0.7566	0.7585	0.7604	0.7623	0.7642	0.7660	40
50	0.7660	0.7679	0.7698	0.7716	0.7735	0.7753	0.7771	39
51	0.7771	0.7790	0.7808	0.7826	0.7844	0.7862	0.7880	38
52	0.7880	0.7898	0.7916	0.7934	0.7951	0.7969	0.7986	37
53	0.7986	0.8004	0.8021	0.8039	0.8056	0.8073	0.8090	36
54	0.8090	0.8107	0.8124	0.8141	0.8158	0.8175	0.8192	35
55	0.8192	0.8208	0.8225	0.8241	0.8258	0.8274	0.8290	34
56	0.8290	0.8307	0.8323	0.8339	0.8355	0.8371	0.8387	33
57	0.8387	0.8403	0.8418	0.8434	0.8450	0.8465	0.8480	32
58	0.8480	0.8496	0.8511	0.8526	0.8542	0.8557	0.8572	31
59	0.8572	0.8587	0.8601	0.8616	0.8631	0.8646	0.8660	30
60	0.8660	0.8675	0.8689	0.8704	0.8718	0.8732	0.8746	29
61	0.8746	0.8760	0.8774	0.8788	0.8802	0.8816	0.8829	28
62	0.8829	0.8843	0.8857	0.8870	0.8884	0.8897	0.8910	27
63	0.8910	0.8923	0.8936	0.8949	0.8962	0.8975	0.8988	26
64	0.8988	0.9001	0.9013	0.9026	0.9038	0.9051	0.9063	25
65	0.9063	0.9075	0.9088	0.9100	0.9112	0.9124	0.9135	24
66	0.9135	0.9147	0.9159	0.9171	0.9182	0.9194	0.9205	23
67	0.9205	0.9216	0.9228	0.9239	0.9250	0.9261	0.9272	22
68	0.9272	0.9283	0.9293	0.9304	0.9315	0.9325	0.9336	21
69	0.9336	0.9346	0.9356	0.9367	0.9377	0.9387	0.9397	20
70	0.9397	0.9407	0.9417	0.9426	0.9436	0.9446	0.9455	19
71	0.9455	0.9465	0.9474	0.9483	0.9492	0.9502	0.9511	18
72	0.9511	0.9520	0.9528	0.9537	0.9546	0.9555	0.9563	17
73	0.9563	0.9572	0.9580	0.9588	0.9596	0.9605	0.9613	16
74	0.9613	0.9621	0.9628	0.9636	0.9644	0.9652	0.9659	15
75	0.9659	0.9667	0.9674	0.9681	0.9689	0.9696	0.9703	14
76	0.9703	0.9710	9.9717	0.9724	0.9730	0.9737	0.9744	13
77	0.9744	0.9750	0.9757	0.9763	0.9769	0.9775	0.9781	12
78	0.9781	0.9787	0.9793	0.9799	0.9805	0.9811	0.9816	11
79	0.9816	0.9822	0.9827	0.9833	0.9838	0.9843	0.9848	10
80	0.9848	0.9853	0.9858	0.9863	0.9868	0.9872	0.9877	9
81	0.9877	0.9881	0.9886	0.9890	0.9894	0.9899	0.9903	8
82	0.9903	0.9907	0.9911	0.9914	0.9918	0.9922	0.9925	7
83	0.9925	0.9929	0.9932	0.9936	0.9939	0.9942	0.9945	6
84	0.9945	0.9948	0.9951	0.9954	0.9957	0.9959	0.9962	5
85	0.9962	0.9964	0.9967	0.9969	0.9971	0.9974	0.9976	4
86	0.9976	0.9978	0.9980	0.9981	0.9983	0.9985	0.9986	3
87	0.9986	0.9988	0.9989	0.9990	0.9992	0.9993	0.9994	2
88	0.9994	0.9995	0.9996	0.9997	0.9997	0.9998	0.9998	1
89	0.9998	0.9999	0.9999	1.0000	1.0000	1.0000	1.0000	0
	60′	50′	40′	30′	20′	10′	0′ Cosine	Deg.

Deg.	Tangent							
	0′	10′	20′	30′	40′	50′	60′	
0	0.0000	0.0029	0.0058	0.0087	0.0116	0.0145	0.0175	89
1	0.0175	0.0204	0.0233	0.0262	0.0291	0.0320	0.0349	88
2	0.0349	0.0378	0.0407	0.0437	0.0466	0.0495	0.0524	87
3	0.0524	0.0553	0.0582	0.0612	0.0641	0.0670	0.0699	86
4	0.0699	0.0729	0.0758	0.0787	0.0816	0.0846	0.0875	85
5	0.0875	0.0904	0.0934	0.0963	0.0992	0.1022	0.1051	84
6	0.1051	0.1080	0.1110	0.1139	0.1169	0.1198	0.1228	83
7	0.1228	0.1257	0.1287	0.1317	0.1346	0.1376	0.1405	82
8	0.1405	0.1435	0.1465	0.1495	0.1524	0.1554	0.1584	81
9	0.1584	0.1614	0.1644	0.1673	0.1703	0.1733	0.1763	80
10	0.1763	0.1793	0.1823	0.1853	0.1883	0.1914	0.1944	79
11	0.1944	0.1974	0.2004	0.2035	0.2065	0.2095	0.2126	78
12	0.2126	0.2156	0.2186	0.2217	0.2247	0.2278	0.2309	77
13	0.2309	0.2339	0.2370	0.2401	0.2432	0.2462	0.2493	76
14	0.2493	0.2524	0.2555	0.2586	0.2617	0.2648	0.2679	75
15	0.2679	0.2711	0.2742	0.2773	0.2805	0.2836	0.2867	74
16	0.2867	0.2899	0.2931	0.2962	0.2994	0.3026	0.3057	73
17	0.3057	0.3089	0.3121	0.3153	0.3185	0.3217	0.3249	72
18	0.3249	0.3281	0.3314	0.3346	0.3378	0.3411	0.3443	71
19	0.3443	0.3476	0.3508	0.3541	0.3574	0.3607	0.3640	70
20	0.3640	0.3673	0.3706	0.3739	0.3772	0.3805	0.3839	69
21	0.3839	0.3872	0.3906	0.3939	0.3973	0.4006	0.4040	68
22	0.4040	0.4074	0.4108	0.4142	0.4176	0.4210	0.4245	67
23	0.4245	0.4279	0.4314	0.4348	0.4383	0.4417	0.4452	66
24	0.4452	0.4487	0.4522	0.4557	0.4592	0.4628	0.4663	65
25	0.4663	0.4699	0.4734	0.4770	0.4806	0.4841	0.4877	64
26	0.4877	0.4913	0.4950	0.4986	0.5022	0.5059	0.5095	63
27	0.5095	0.5132	0.5169	0.5206	0.5243	0.5280	0.5317	62
28	0.5317	0.5354	0.5392	0.5430	0.5467	0.5505	0.5543	61
29	0.5543	0.5581	0.5619	0.5658	0.5696	0.5735	0.5774	60
30	0.5774	0.5812	0.5851	0.5890	0.5930	0.5969	0.6009	59
31	0.6009	0.6048	0.6088	0.6128	0.6168	0.6208	0.6249	58
32	0.6249	0.6289	0.6330	0.6371	0.6412	0.6453	0.6494	57
33	0.6494	0.6536	0.6577	0.6619	0.6661	0.6703	0.6745	56
34	0.6745	0.6787	0.6830	0.6873	0.6916	0.6959	0.7002	55
35	0.7002	0.7046	0.7089	0.7133	0.7177	0.7221	0.7265	54
36	0.7265	0.7310	0.7355	0.7400	0.7445	0.7490	0.7536	53
37	0.7536	0.7581	0.7627	0.7673	0.7720	0.7766	0.7813	52
38	0.7813	0.7860	0.7907	0.7954	0.8002	0.8050	0.8098	51
39	0.8098	0.8146	0.8195	0.8243	0.8292	0.8342	0.8391	50
40	0.8391	0.8441	0.8491	0.8541	0.8591	0.8642	0.8693	49
41	0.8693	0.8744	0.8796	0.8847	0.8899	0.8952	0.9004	48
42	0.9004	0.9057	0.9110	0.9163	0.9217	0.9271	0.9325	47
43	0.9325	0.9380	0.9435	0.9490	0.9545	0.9601	0.9657	46
44	0.9657	0.9713	0.9770	0.9827	0.9884	0.9942	1.0000	45
	60′	50′	40′	30′	20′	10′	0′	Deg.
							Cotangent	

Deg.	Tangent							
	0′	10′	20′	30′	40′	50′	60′	
45	1.000	1.006	1.012	1.018	1.024	1.030	1.036	44
46	1.036	1.042	1.048	1.054	1.060	1.066	1.072	43
47	1.072	1.079	1.085	1.091	1.098	1.104	1.111	42
48	1.111	1.117	1.124	1.130	1.137	1.144	1.150	41
49	1.150	1.157	1.164	1.171	1.178	1.185	1.192	40
50	1.192	1.199	1.206	1.213	1.220	1.228	1.235	39
51	1.235	1.242	1.250	1.257	1.265	1.272	1.280	38
52	1.280	1.288	1.295	1.303	1.311	1.319	1.327	37
53	1.327	1.335	1.343	1.351	1.360	1.368	1.376	36
54	1.376	1.385	1.393	1.402	1.411	1.419	1.428	35
55	1.428	1.437	1.446	1.455	1.464	1.473	1.483	34
56	1.483	1.492	1.501	1.511	1.520	1.530	1.540	33
57	1.540	1.550	1.560	1.570	1.580	1.590	1.600	32
58	1.600	1.611	1.621	1.632	1.643	1.653	1.664	31
59	1.664	1.675	1.686	1.698	1.709	1.720	1.732	30
60	1.732	1.744	1.756	1.767	1.780	1.792	1.804	29
61	1.804	1.816	1.829	1.842	1.855	1.868	1.881	28
62	1.881	1.894	1.907	1.921	1.935	1.949	1.963	27
63	1.963	1.977	1.991	2.006	2.020	2.035	2.050	26
64	2.050	2.066	2.081	2.097	2.112	2.128	2.145	25
65	2.145	2.161	2.177	2.194	2.211	2.229	2.246	24
66	2.246	2.264	2.282	2.300	2.318	2.337	2.356	23
67	2.356	2.375	2.394	2.414	2.434	2.455	2.475	22
68	2.475	2.496	2.517	2.539	2.560	2.583	2.605	21
69	2.605	2.628	2.651	2.675	2.699	2.723	2.747	20
70	2.747	2.773	2.798	2.824	2.850	2.877	2.904	19
71	2.904	2.932	2.960	2.989	3.018	3.047	3.078	18
72	3.078	3.108	3.140	3.172	3.204	3.237	3.271	17
73	3.271	3.305	3.340	3.376	3.412	3.450	3.487	16
74	3.487	3.526	3.566	3.606	3.647	3.689	3.732	15
75	3.732	3.776	3.821	3.867	3.914	3.962	4.011	14
76	4.011	4.061	4.113	4.165	4.219	4.275	4.331	13
77	4.331	4.390	4.449	4.511	4.574	4.638	4.705	12
78	4.705	4.773	4.843	4.915	4.989	5.066	5.145	11
79	5.145	5.226	5.309	5.396	5.485	5.576	5.671	10
80	5.671	5.769	5.871	5.976	6.084	6.197	6.314	9
81	6.314	6.435	6.561	6.691	6.827	6.968	7.115	8
82	7.115	7.269	7.429	7.596	7.770	7.953	8.144	7
83	8.144	8.345	8.556	8.777	9.010	9.255	9.514	6
84	9.514	9.788	10.08	10.39	10.71	11.06	11.43	5
85	11.43	11.83	12.25	12.71	13.20	13.73	14.30	4
86	14.30	14.92	15.60	16.35	17.17	18.07	19.08	3
87	19.08	20.21	21.47	22.90	24.54	26.43	28.64	2
88	28.64	31.24	34.37	38.19	42.96	49.10	57.29	1
89	57.29	68.75	85.94	114.6	171.9	343.8	∞	0
	60′	50′	40′	30′	20′	10′	0′	Deg.
						Cotangent		

Sines

Deg.	0′	10′	20′	30′	40′	50′	60′	
0	Inf. Neg.	7.4637	7.7648	7.9408	8.0658	8.1627	8.2419	89
1	8.2419	8.3088	8.3668	8.4179	8.4637	8.5050	8.5428	88
2	8.5428	8.5776	8.6097	8.6397	8.6677	8.6940	8.7188	87
3	8.7188	8.7423	8.7645	8.7857	8.8059	8.8251	8.8436	86
4	8.8436	8.8613	8.8783	8.8946	8.9104	8.9256	8.9403	85
5	8.9403	8.9545	8.9682	8.9816	8.9945	9.0070	9.0192	84
6	9.0192	9.0311	9.0426	9.0539	9.0648	9.0755	9.0859	83
7	9.0859	9.0961	9.1060	9.1157	9.1252	9.1345	9.1436	82
8	9.1436	9.1525	9.1612	9.1697	9.1781	9.1863	9.1943	81
9	9.1943	9.2022	9.2100	9.2176	9.2251	9.2324	9.2397	80
10	9.2397	9.2468	9.2538	9.2606	9.2674	9.2740	9.2806	79
11	9.2806	9.2870	9.2934	9.2997	9.3058	9.3119	9.3179	78
12	9.3179	9.3238	9.3296	9.3353	9.3410	9.3466	9.3521	77
13	9.3521	9.3575	9.3629	9.3682	9.3734	9.3786	9.3837	76
14	9.3837	9.3887	9.3937	9.3986	9.4035	9.4083	9.4130	75
15	9.4130	9.4177	9.4223	9.4269	9.4314	9.4359	9.4403	74
16	9.4403	9.4447	9.4491	9.4533	9.4576	9.4618	9.4659	73
17	9.4659	9.4700	9.4741	9.4781	9.4821	9.4861	9.4900	72
18	9.4900	9.4939	9.4977	9.5015	9.5052	9.5090	9.5126	71
19	9.5126	9.5163	9.5199	9.5235	9.5270	9.5306	9.5341	70
20	9.5341	9.5375	9.5409	9.5443	9.5477	9.5510	9.5543	69
21	9.5543	9.5576	9.5609	9.5641	9.5673	9.5704	9.5736	68
22	9.5736	9.5767	9.5798	9.5828	9.5859	9.5889	9.5919	67
23	9.5919	9.5948	9.5978	9.6007	9.6036	9.6065	9.6093	66
24	9.6093	9.6121	9.6149	9.6177	9.6205	9.6232	9.6259	65
25	9.6259	9.6286	9.6313	9.6340	9.6366	9.6392	9.6418	64
26	9.6418	9.6444	9.6470	9.6495	9.6521	9.6546	9.6570	63
27	9.6570	9.6595	9.6620	9.6644	9.6668	9.6692	9.6716	62
28	9.6716	9.6740	9.6763	9.6787	9.6810	9.6833	9.6856	61
29	9.6856	9.6878	9.6901	9.6923	9.6946	9.6968	9.6990	60
30	9.6990	9.7012	9.7033	9.7055	9.7076	9.7097	9.7118	59
31	9.7118	9.7139	9.7160	9.7181	9.7201	9.7222	9.7242	58
32	9.7242	9.7262	9.7282	9.7302	9.7322	9.7342	9.7361	57
33	9.7361	9.7380	9.7400	9.7419	9.7438	9.7457	9.7476	56
34	9.7476	9.7494	9.7513	9.7531	9.7550	9.7568	9.7586	55
35	9.7586	9.7604	9.7622	9.7640	9.7657	9.7675	9.7692	54
36	9.7692	9.7710	9.7727	9.7744	9.7761	9.7778	9.7795	53
37	9.7795	9.7811	9.7828	9.7844	9.7861	9.7877	9.7893	52
38	9.7893	9.7910	9.7926	9.7941	9.7957	9.7973	9.7989	51
39	9.7989	9.8004	9.8020	9.8035	9.8050	9.8066	9.8081	50
40	9.8081	9.8096	9.8111	9.8125	9.8140	9.8155	9.8169	49
41	9.8169	9.8184	9.8198	9.8213	9.8227	9.8241	9.8255	48
42	9.8255	9.8269	9.8283	9.8297	9.8311	9.8324	9.8338	47
43	9.8338	9.8351	9.8365	9.8378	9.8391	9.8405	9.8418	46
44	9.8418	9.8431	9.8444	9.8457	9.8469	9.8482	9.8495	45
	60′	50′	40′	30′	20′	10′	0′	Deg.

Cosines

Sines

Deg.	0′	10′	20′	30′	40′	50′	60′	
45	9.8495	9.8507	9.8520	9.8532	9.8545	9.8557	9.8569	44
46	9.8569	9.8582	9.8594	9.8606	9.8618	9.8629	9.8641	43
47	9.8641	9.8653	9.8665	9.8676	9.8688	9.8699	9.8711	42
48	9.8711	9.8722	9.8733	9.8745	9.8756	9.8767	9.8778	41
49	9.8778	9.8789	9.8800	9.8810	9.8821	9.8832	9.8843	40
50	9.8843	9.8853	9.8864	9.8874	9.8884	9.8895	9.8905	39
51	9.8905	9.8915	9.8925	9.8935	9.8945	9.8955	9.8965	38
52	9.8965	9.8975	9.8985	9.8995	9.9004	9.9014	9.9023	37
53	9.9023	9.9033	9.9042	9.9052	9.9061	9.9070	9.9080	36
54	9.9080	9.9089	9.9098	9.9107	9.9116	9.9125	9.9134	35
55	9.9134	9.9142	9.9151	9.9160	9.9169	9.9177	9.9186	34
56	9.9186	9.9194	9.9203	9.9211	9.9219	9.9228	9.9236	33
57	9.9236	9.9244	9.9252	9.9260	9.9268	9.9276	9.9284	32
58	9.9284	9.9292	9.9300	9.9308	9.9315	9.9323	9.9331	31
59	9.9331	9.9338	9.9346	9.9353	9.9361	9.9368	9.9375	30
60	9.9375	9.9383	9.9390	9.9397	9.9404	9.9411	9.9418	29
61	9.9418	9.9425	9.9432	9.9439	9.9446	9.9453	9.9459	28
62	9.9459	9.9466	9.9473	9.9479	9.9486	9.9492	9.9499	27
63	9.9499	9.9505	9.9512	9.9518	9.9524	9.9530	9.9537	26
64	9.9537	9.9543	9.9549	9.9555	9.9561	9.9567	9.9573	25
65	9.9573	9.9579	9.9584	9.9590	9.9596	9.9602	9.9607.	24
66	9.9607	9.9613	9.9618	9.9624	9.9629	9.9635	9.9640	23
67	9.9640	9.9646	9.9651	9.9656	9.9661	9.9667	9.9672	22
68	9.9672	9.9677	9.9682	9.9687	9.9692	9.9697	9.9702	21
69	9.9702	9.9706	9.9711	9.9716	9.9721	9.9725	9.9730	20
70	9.9730	9.9734	9.9739	9.9743	9.9748	9.9752	9.9757	19
71	9.9757	9.9761	9.9765	9.9770	9.9774	9.9778	9.9782	18
72	9.9782	9.9786	9.9790	9.9794	9.9798	9.9802	9.9806	17
73	9.9806	9.9810	9.9814	9.9817	9.9821	9.9825	9.9828	16
74	9.9828	9.9832	9.9836	9.9839	9.9843	9.9846	9.9849	15
75	9.9849	9.9853	9.9856	9.9859	9.9863	9.9866	9.9869	14
76	9.9869	9.9872	9.9875	9.9878	9.9881	9.9884	9.9887	13
77	9.9887	9.9890	9.9893	9.9896	9,9899	9.9901	9.9904	12
78	9.9904	9.9907	9.9909	9.9912	9.9914	9.9917	9.9919	11
79	9.9919	9.9922	9.9924	9.9927	9.9929	9.9931	9.9934	10
80	9.9934	9.9936	9.9938	9.9940	9.9942	9.9944	9.9946	9
81	9.9946	9.9948	9.9950	9.9952	9.9954	9.9956	9.9958	8
82	9.9958	9.9959	9.9961	9.9963	9.9964	9.9966	9.9968	7
83	9.9968	9.9969	9.9971	9.9972	9.9973	9.9975	9.9976	6
84	9.9976	9.9977	9.9979	9.9980	9.9981	9.9982	9.9983	5
85	9.9983	9.9985	9.9986	9.9987	9.9988	9.9989	9.9989	4
86	9.9989	9.9990	9.9991	9.9992	9.9993	9.9993	9.9994	3
87	9.9994	9.9995	9.9995	9.9996	9.9996	9.9997	9.9997	2
88	9.9997	9.9998	9.9998	9.9999	9.9999	9.9999	9.9999	1
89	9.9999	10.000	10.000	10.000	10.000	10.000	10.0000	0
	60′	50′	40′	30′	20′	10′	0′	Deg.

Cosines

Tangents

Deg.	0′	10′	20′	30′	40′	50′	60′	
0	Inf. Neg.	7.4637	7.7648	7.9409	8.0658	8.1627	8.2419	89
1	8.2419	8.3089	8.3669	8.4181	8.4638	8.5053	8.5431	88
2	8.5431	8.5779	8.6101	8.6401	8.6682	8.6945	8.7194	87
3	8.7194	8.7429	8.7652	8.7865	8.8067	8.8261	8.8446	86
4	8.8446	8.8624	8.8795	8.8960	8.9118	8.9272	8.9420	85
5	8.9420	8.9563	8.9701	8.9836	8.9966	9.0093	9.0216	84
6	9.0216	9.0336	9.0453	9.0567	9.0678	9.0786	9.0891	83
7	9.0891	9.0995	9.1096	9.1194	9.1291	9.1385	9.1478	82
8	9.1478	9.1569	9.1658	9.1745	9.1831	9.1915	9.1997	81
9	9.1997	9.2078	9.2158	9.2236	9.2313	9.2389	9.2463	80
10	9.2463	9.2536	9.2609	9.2680	9.2750	9.2819	9.2887	79
11	9.2887	9.2953	9.3020	9.3085	9.3149	9.3212	9.3275	78
12	9.3275	9.3336	9.3397	9.3458	9.3517	9.3576	9.3634	77
13	9.3634	9.3691	9.3748	9.3804	9.3859	9.3914	9.3968	76
14	9.3968	9.4021	9.4074	9.4127	9.4178	9.4230	9.4281	75
15	9.4281	9.4331	9.4381	9.4430	9.4479	9.4527	9.4575	74
16	9.4575	9.4622	9.4669	9.4716	9.4762	9.4808	9.4853	73
17	9.4853	9.4898	9.4943	9.4987	9.5031	9.5075	9.5118	72
18	9.5118	9.5161	9.5203	9.5245	9.5287	9.5329	9.5370	71
19	9.5370	9.5411	9.5451	9.5491	9.5531	9.5571	9.5611	70
20	9.5611	9.5650	9.5689	9.5727	9.5766	9.5804	9.5842	69
21	9.5842	9.5879	9.5917	9.5954	9.5991	9.6028	9.6064	68
22	9.6064	9.6100	9.6136	9.6172	9.6208	9.6243	9.6279	67
23	9.6279	9.6314	9.6348	9.6383	9.6417	9.6452	9.6486	66
24	9.6486	9.6520	9.6553	9.6587	9.6620	9.6654	9.6687	65
25	9.6687	9.6720	9.6752	9.6785	9.6817	9.6850	9.6882	64
26	9.6882	9.6914	9.6946	9.6977	9.7009	9.7040	9.7072	63
27	9.7072	9.7103	9.7134	9.7165	9.7196	9.7226	9.7257	62
28	9.7257	9.7287	9.7317	9.7348	9.7378	9.7408	9.7438	61
29	9.7438	9.7467	9.7497	9.7526	9.7556	9.7585	9.7614	60
30	9.7614	9.7644	9.7673	9.7701	9.7730	9.7759	9.7788	59
31	9.7788	9.7816	9.7845	9.7873	9.7902	9.7930	9.7958	58
32	9.7958	9.7986	9.8014	9.8042	9.8070	9.8097	9.8125	57
33	9.8125	9.8153	9.8180	9.8208	9.8235	9.8263	9.8290	56
34	9.8290	9.8317	9.8344	9.8371	9.8398	9.8425	9.8452	55
35	9.8452	9.8479	9.8506	9.8533	9.8559	9.8586	9.8613	54
36	9.8613	9.8639	9.8666	9.8692	9.8718	9.8745	9.8771	53
37	9.8771	9.8797	9.8824	9.8850	9.8876	9.8902	9.8928	52
38	9.8928	9.8954	9.8980	9.9006	9.9032	9.9058	9.9084	51
39	9.9084	9.9110	9.9135	9.9161	9.9187	9.9212	9.9238	50
40	9.9238	9.9264	9.9289	9.9315	9.9341	9.9366	9.9392	49
41	9.9392	9.9417	9.9443	9.9468	9.9494	9.9519	9.9544	48
42	9.9544	9.9570	9.9595	9.9621	9.9646	9.9671	9.9697	47
43	9.9697	9.9722	9.9747	9.9772	9.9798	9.9823	9.9848	46
44	9.9848	9.9874	9.9899	9.9924	9.9949	9.9975	10.0000	45
	60′	50′	40′	30′	20′	10′	0′	Deg.

Cotangents

Tangents

Deg.	0′	10′	20′	30′	40′	50′	60′	
45	0.0000	0.0025	0.0051	0.0076	0.0101	0.0126	0.0152	44
46	0.0152	0.0177	0.0202	0.0228	0.0253	0.0278	0.0303	43
47	0.0303	0.0329	0.0354	0.0379	0.0405	0.0430	0.0456	42
48	0.0456	0.0481	0.0506	0.0532	0.0557	0.0583	0.0608	41
49	0.0608	0.0634	0.0659	0.0685	0.0711	0.0736	0.0762	40
50	0.0762	0.0788	0.0813	0.0839	0.0865	0.0890	0.0916	39
51	0.0916	0.0942	0.0968	0.0994	0.1020	0.1046	0.1072	38
52	0.1072	0.1098	0.1124	0.1150	0.1176	9.1203	0.1229	37
53	0.1229	0.1255	0.1282	0.1308	0.1334	0.1361	0.1387	36
54	0.1387	0.1414	0.1441	0.1467	0.1494	0.1521	0.1548	35
55	0.1548	0.1575	0.1602	0.1629	0.1656	0.1683	0.1710	34
56	0.1710	0.1737	0.1765	0.1792	0.1820	0.1847	0.1875	33
57	0.1875	0.1903	0.1930	0.1958	0.1986	0.2014	0.2042	32
58	0.2042	0.2070	0.2098	0.2127	0.2155	0.2184	0.2212	31
59	0.2212	0.2241	0.2270	0.2299	0.2327	0.2356	0.2386	30
60	0.2386	0.2415	0.2444	0.2474	0.2503	0.2533	0.2562	29
61	0.2562	0.2592	0.2622	0.2652	0.2683	0.2713	0.2743	28
62	0.2743	0.2774	0.2804	0.2835	0.2866	0.2897	0.2928	27
63	0.2928	0.2960	0.2991	0.3023	0.3054	0.3086	0.3118	26
64	0.3118	0.3150	0.3183	0.3215	0.3248	0.3280	0.3313	25
65	0.3313	0.3346	0.3380	0.3413	0.3447	0.3480	0.3514	24
66	0.3514	0.3548	0.3583	0.3617	0.3652	0.3686	0.3721	23
67	0.3721	0.3757	0.3792	0.3828	0.3864	0.3900	0.3936	22
68	0.3936	0.3972	0.4009	0.4046	0.4083	0.4121	0.4158	21
69	0.4158	0.4196	0.4234	0.4273	0.4311	0.4350	0.4389	20
70	0.4389	0.4429	0.4469	0.4509	0.4549	0.4589	0.4630	19
71	0.4630	0.4671	0.4713	0.4755	0.4797	0.4839	0.4882	18
72	0.4882	0.4925	0.4969	0.5013	0.5057	0.5102	0.5147	17
73	0.5147	0.5192	0.5238	0.5284	0.5331	0.5378	0.5425	16
74	0.5425	0.5473	0.5521	0.5570	0.5619	0.5669	0.5719	15
75	0.5719	0.5770	0.5822	0.5873	0.5926	0.5979	0.6032	14
76	0.6032	0.6086	0.6141	0.6196	0.6252	0.6309	0.6366	13
77	0.6366	0.6424	0.6483	0.6542	0.6603	0.6664	0.6725	12
78	0.6725	0.6788	0.6851	0.6915	0.6980	0.7047	0.7113	11
79	0.7113	0.7181	0.7250	0.7320	0.7391	0.7464	0.7537	10
80	0.7537	0.7611	0.7687	0.7764	0.7842	0.7922	0.8003	9
81	0.8003	0.8085	0.8169	0.8255	0.8342	0.8431	0.8522	8
82	0.8522	0.8615	0.8709	0.8806	0.8904	0.9008	0.9109	7
83	0.9109	0.9214	0.9322	0.9433	0.9547	0.9664	0.9784	6
84	0.9784	0.9907	1.0034	1.0164	1.0299	1.0437	1.0580	5
85	1.0580	1.0728	1.0882	1.1040	1.1205	1.1376	1.1554	4
86	1.1554	1.1739	1.1933	1.2135	1.2348	1.2571	1.2806	3
87	1.2806	1.3055	1.3318	1.3599	1.3899	1.4221	1.4569	2
88	1.4569	1.4947	1.5362	1.5819	1.6331	1.6911	1.7581	1
89	1.7581	1.8373	1.9342	2.0591	2.2352	2.5363	Infinite	0
	60′	50′	40′	30′	20′	10′	0′	Deg.

Cotangents

Sine	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9		Cosine
		.9	.8	.7	.6	.5	.4	.3	.2	.1	.0	
											0.000000	90°
0°	0.000000	001745	003491	005236	006981	008727	010472	012217	013962	015707	017452	89
1	01745	01920	02094	02269	02443	02618	02792	02967	03141	03316	03490	88
2	03490	03664	03839	04013	04188	04362	04536	04711	04885	05059	05234	87
3	05234	05408	05582	05756	05931	06105	06279	06453	06627	06802	06976	86
4	06976	07150	07324	07498	07672	07846	08020	08194	08368	08542	08716	85
5	08716	08889	09063	09237	09411	09585	09758	09932	10106	10279	10453	84
6	10453	10626	10800	10973	11147	11320	11494	11667	11840	12014	12187	83
7	12187	12360	12533	12706	12880	13053	13226	13399	13572	13744	13917	82
8	13917	14090	14263	14436	14608	14781	14954	15126	15299	15471	15643	81
9	15643	15816	15988	16160	16333	16505	16677	16849	17021	17193	0.17365	80°
10°	0.1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	1908	79
11	1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	2079	78
12	2079	2096	2113	2130	2147	2164	2181	2198	2215	2233	2250	77
13	2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	2419	76
14	2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	2588	75
15	2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	2756	74
16	2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	2924	73
17	2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3090	72
18	3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3256	71
19	3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	0.3420	70°
20°	0.3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3584	69
21	3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3746	68
22	3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3907	67
23	3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	4067	66
24	4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	4226	65
25	4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	4384	64
26	4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	4540	63
27	4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	4695	62
28	4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	4848	61
29	4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	0.5000	60°
30°	0.5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	5150	59
31	5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	5299	58
32	5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	5446	57
33	5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	5592	56
34	5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	5736	55
35	5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	5878	54
36	5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	6018	53
37	6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	6157	52
38	6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	6293	51
39	6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	0.6428	50°
40°	0.6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	6561	49
41	6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	6691	48
42	6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	6820	47
43	6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	6947	46
44	6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	0.7071	45°
45°	0.7071											

Sine	0	.1	.2	.3	.4	.5	.6	.7	.8	.9		Cosine
		.9	.8	.7	.6	.5	.4	.3	.2	.1	.0	
											0.7071	45°
45°	0.7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	7193	44
46	7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	7314	43
47	7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	7431	42
48	7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	7547	41
49	7547	7559	7570	7581	7593	7604	7615	7627	7638	7649	7660	40°
50°	0.7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	0.7771	39
51	7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	7880	38
52	7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	7986	37
53	7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	8090	36
54	8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	8192	35
55	8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	8290	34
56	8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	8387	33
57	8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	8480	32
58	8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	8572	31
59	8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	0.8660	30°
60°	0.8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	8746	29
61	8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	8829	28
62	8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	8910	27
63	8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	8988	26
64	8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	9063	25
65	9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	9135	24
66	9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	9205	23
67	9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	9272	22
68	9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	9336	21
69	9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	0.9397	20°
70°	0.9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	9455	19
71	9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	9511	18
72	9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	9563	17
73	9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	9613	16
74	9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	9659	15
75	9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	9703	14
76	9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	9744	13
77	9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	9781	12
78	9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	9816	11
79	9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	0.9848	10°
80°	0.9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	9877	9
81	9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	9903	8
82	9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	9925	7
83	9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	9945	6
84	9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	9962	5
85	9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	9976	4
86	9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	9986	3
87	9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	9994	2
88	9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	0.9998	1
89	0.9998	9999	9999	9999	9999	0000	0000	0000	0000	0000	1.0000	0°
90°	1.0000											

Tangent	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9		Cotangent
		.9	.8	.7	.6	.5	.4	.3	.2	.1	.0	
											0.000000	90°
0°	0.000000	001745	003491	005236	006981	008727	010472	012218	013964	015709	017455	89
1	01746	01920	02095	02269	02444	02619	02793	02968	03143	03317	03492	88
2	03492	03667	03842	04016	04191	04366	04541	04716	04891	05066	05241	87
3	05241	05416	05591	05766	05941	06116	06291	06467	06642	06817	06993	86
4	06993	07168	07344	07519	07695	07870	08046	08221	08397	08573	08749	85
5	08749	08925	09101	09277	09453	09629	09805	09981	10158	10334	10510	84
6	10510	10687	10863	11040	11217	11394	11570	11747	11924	12101	12278	83
7	12278	12456	12633	12810	12988	13165	13343	13521	13698	13876	14054	82
8	14054	14232	14410	14588	14767	14945	15124	15302	15481	15660	15838	81
9	15838	16017	16196	16376	16555	16734	16914	17093	17273	17453	0.17633	80°
10°	0.1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	1944	79
11	1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	2126	78
12	2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	2309	77
13	2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	2493	76
14	2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	2679	75
15	2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	2867	74
16	2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3057	73
17	3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3249	72
18	3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3443	71
19	3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	0.3640	70°
20°	0.3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3839	69
21	3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	4040	68
22	4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	4245	67
23	4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	4452	66
24	4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4663	65
25	4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4877	64
26	4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	5095	63
27	5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	5317	62
28	5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	5543	61
29	5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	0.5774	60°
30°	0.5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	6009	59
31	6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	6249	58
32	6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	6494	57
33	6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	6745	56
34	6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	7002	55
35	7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	7265	54
36	7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	7536	53
37	7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	7813	52
38	7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	8098	51
39	8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	0.8391	50°
40°	0.8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	8693	49
41	8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	9004	48
42	9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	9325	47
43	9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	0.9657	46
44	0.9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	1.0000	45°
45°	1.0000											

Tangent	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9		Cotangent
		.9	.8	.7	.6	.5	.4	.3	.2	.1	.0	
											1.0000	45°
45°	1.0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	0355	44
46	0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	0724	43
47	0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	1106	42
48	1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	1504	41
49	1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	1.1918	40°
50°	1.1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	2349	39
51	2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	2799	38
52	2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	3270	37
53	3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	3764	36
54	3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	1.4281	35
55	1.4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	4826	34
56	4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	5399	33
57	5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	6003	32
58	6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	6643	31
59	1.6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	1.7321	30°
60°	1.732	1.739	1.746	1.753	1.760	1.767	1.775	1.782	1.789	1.797	1.804	29
61	1.804	1.811	1.819	1.827	1.834	1.842	1.849	1.857	1.865	1.873	1.881	28
62	1.881	1.889	1.897	1.905	1.913	1.921	1.929	1.937	1.946	1.954	1.963	27
63	1.963	1.971	1.980	1.988	1.997	2.006	2.014	2.023	2.032	2.041	2.050	26
64	2.050	2.059	2.069	2.078	2.087	2.097	2.106	2.116	2.125	2.135	2.145	25
65	2.145	2.154	2.164	2.174	2.184	2.194	2.204	2.215	2.225	2.236	2.246	24
66	2.246	2.257	2.267	2.278	2.289	2.300	2.311	2.322	2.333	2.344	2.356	23
67	2.356	2.367	2.379	2.391	2.402	2.414	2.426	2.438	2.450	2.463	2.475	22
68	2.475	2.488	2.500	2.513	2.526	2.539	2.552	2.565	2.578	2.592	2.605	21
69	2.605	2.619	2.633	2.646	2.660	2.675	2.689	2.703	2.718	2.733	2.747	20°
70°	2.747	2.762	2.778	2.793	2.808	2.824	2.840	2.856	2.872	2.888	2.904	19
71	2.904	2.921	2.937	2.954	2.971	2.989	3.006	3.024	3.042	3.060	3.078	18
72	3.078	3.096	3.115	3.133	3.152	3.172	3.191	3.211	3.230	3.251	3.271	17
73	3.271	3.291	3.312	3.333	3.354	3.376	3.398	3.420	3.442	3.465	3.487	16
74	3.487	3.511	3.534	3.558	3.582	3.606	3.630	3.655	3.681	3.706	3.732	15
75	3.732	3.758	3.785	3.812	3.839	3.867	3.895	3.923	3.952	3.981	4.011	14
76	4.011	4.041	4.071	4.102	4.134	4.165	4.198	4.230	4.264	4.297	4.331	13
77	4.331	4.366	4.402	4.437	4.474	4.511	4.548	4.586	4.625	4.665	4.705	12
78	4.705	4.745	4.787	4.829	4.872	4.915	4.959	5.005	5.050	5.097	5.145	11
79	5.145	5.193	5.242	5.292	5.343	5.396	5.449	5.503	5.558	5.614	5.671	10°
80°	5.671	5.730	5.789	5.850	5.912	5.976	6.041	6.107	6.174	6.243	6.314	9
81	6.314	6.386	6.460	6.535	6.612	6.691	6.772	6.855	6.940	7.026	7.115	8
82	7.115	7.207	7.300	7.396	7.495	7.596	7.700	7.806	7.916	8.028	8.144	7
83	8.144	8.264	8.386	8.513	8.643	8.777	8.915	9.058	9.205	9.357	9.514	6
84	9.514	9.677	9.845	10.019	10.199	10.385	10.579	10.780	10.988	11.205	11.430	5
85	11.430	11.664	11.909	12.163	12.429	12.706	12.996	13.300	13.617	13.951	14.301	4
86	14.301	14.669	15.056	15.464	15.895	16.350	16.832	17.343	17.886	18.464	19.081	3
87	19.08	19.74	20.45	21.20	22.02	22.90	23.86	24.90	26.03	27.27	28.64	2
88	28.64	30.14	31.82	33.69	35.80	38.19	40.92	44.07	47.74	52.08	57.29	1
89	57.29	63.66	71.62	81.85	95.49	114.59	143.24	191.0	286.5	573.0	∞	0°
90°	∞											

Log Sin **Log Cos**

	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9		
		.9	.8	.7	.6	.5	.4	.3	.2	.1	.0	
											−∞	90°
0°	−∞	$\bar{3}$.2419	5429	7190	8439	9408	∫0200	0870	1450	1961	$\bar{2}$.2419	89
1	$\bar{2}$.2419	2832	3210	3558	3880	4179	4459	4723	4971	5206	5428	88
2	5428	5640	5842	6035	6220	6397	6567	6731	6889	7041	7188	87
3	7188	7330	7468	7602	7731	7857	7979	8098	8213	8326	8436	86
4	8436	8543	8647	8749	8849	8946	9042	9135	9226	9315	$\bar{2}$.9403	85
5	$\bar{2}$.9403	9489	9573	9655	9736	9816	9894	9970	∫0046	0120	$\bar{1}$.0192	84
6	$\bar{1}$.0192	0264	0334	0403	0472	0539	0605	0670	0734	0797	0859	83
7	0859	0920	0981	1040	1099	1157	1214	1271	1326	1381	1436	82
8	1436	1489	1542	1594	1646	1697	1747	1797	1847	1895	1943	81
9	1943	1991	2038	2085	2131	2176	2221	2266	2310	2353	$\bar{1}$.2397	80°
10°	$\bar{1}$.2397	2439	2482	2524	2565	2606	2647	2687	2727	2767	2806	79
11	2806	2845	2883	2921	2959	2997	3034	3070	3107	3143	3179	78
12	3179	3214	3250	3284	3319	3353	3387	3421	3455	3488	3521	77
13	3521	3554	3586	3618	3650	3682	3713	3745	3775	3806	3837	76
14	3837	3867	3897	3927	3957	3986	4015	4044	4073	4102	4130	75
15	4130	4158	4186	4214	4242	4269	4296	4323	4350	4377	4403	74
16	4403	4430	4456	4482	4508	4533	4559	4584	4609	4634	4659	73
17	4659	4684	4709	4733	4757	4781	4805	4829	4853	4876	4900	72
18	4900	4923	4946	4969	4992	5015	5037	5060	5082	5104	5126	71
19	5126	5148	5170	5192	5213	5235	5256	5278	5299	5320	$\bar{1}$.5341	70°
20°	$\bar{1}$.5341	5361	5382	5402	5423	5443	5463	5484	5504	5523	5543	69
21	5543	5563	5583	5602	5621	5641	5660	5679	5698	5717	5736	68
22	5736	5754	5773	5792	5810	5828	5847	5865	5883	5901	5919	67
23	5919	5937	5954	5972	5990	6007	6024	6042	6059	6076	6093	66
24	6093	6110	6127	6144	6161	6177	6194	6210	6227	6243	6259	65
25	6259	6276	6292	6308	6324	6340	6356	6371	6387	6403	6418	64
26	6418	6434	6449	6465	6480	6495	6510	6526	6541	6556	6570	63
27	6570	6585	6600	6615	6629	6644	6659	6673	6687	6702	6716	62
28	6716	6730	6744	6759	6773	6787	6801	6814	6828	6842	6856	61
29	6856	6869	6883	6896	6910	6923	6937	6950	6963	6977	$\bar{1}$.6990	60°
30°	$\bar{1}$.6990	7003	7016	7029	7042	7055	7068	7080	7093	7106	7118	59
31	7118	7131	7144	7156	7168	7181	7193	7205	7218	7230	7242	58
32	7242	7254	7266	7278	7290	7302	7314	7326	7338	7349	7361	57
33	7361	7373	7384	7396	7407	7419	7430	7442	7453	7464	7476	56
34	7476	7487	7498	7509	7520	7531	7542	7553	7564	7575	7586	55
35	7586	7597	7607	7618	7629	7640	7650	7661	7671	7682	7692	54
36	7692	7703	7713	7723	7734	7744	7754	7764	7774	7785	7795	53
37	7795	7805	7815	7825	7835	7844	7854	7864	7874	7884	7893	52
38	7893	7903	7913	7922	7932	7941	7951	7960	7970	7979	7989	51
39	7989	7998	8007	8017	8026	8035	8044	8053	8063	8072	$\bar{1}$.8081	50°
40°	$\bar{1}$.8081	8090	8099	8108	8117	8125	8134	8143	8152	8161	8169	49
41	8169	8178	8187	8195	8204	8213	8221	8230	8238	8247	8255	48
42	8255	8264	8272	8280	8289	8297	8305	8313	8322	8330	8338	47
43	8338	8346	8354	8362	8370	8378	8386	8394	8402	8410	8418	46
44	8418	8426	8433	8441	8449	8457	8464	8472	8480	8487	$\bar{1}$.8495	45°
45°	$\bar{1}$.8495											

Log Sin **Log Cos**

	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9		
		.9	.8	.7	.6	.5	.4	.3	.2	.1	.0	
											$\bar{1}$.8495	45°
45°	$\bar{1}$.8495	8502	8510	8517	8525	8532	8540	8547	8555	8562	8569	44
46	8569	8577	8584	8591	8598	8606	8613	8620	8627	8634	8641	43
47	8641	8648	8655	8662	8669	8676	8683	8690	8697	8704	8711	42
48	8711	8718	8724	8731	8738	8745	8751	8758	8765	8771	8778	41
49	8778	8784	8791	8797	8804	8810	8817	8823	8830	8836	$\bar{1}$.8843	40°
50°	$\bar{1}$.8843	8849	8855	8862	8868	8874	8880	8887	8893	8899	8905	39
51	8905	8911	8917	8923	8929	8935	8941	8947	8953	8959	8965	38
52	8965	8971	8977	8983	8989	8995	9000	9006	9012	9018	9023	37
53	9023	9029	9035	9041	9046	9052	9057	9063	9069	9074	9080	36
54	9080	9085	9091	9096	9101	9107	9112	9118	9123	9128	9134	35
55	9134	9139	9144	9149	9155	9160	9165	9170	9175	9181	9186	34
56	9186	9191	9196	9201	9206	9211	9216	9221	9226	9231	9236	33
57	9236	9241	9246	9251	9255	9260	9265	9270	9275	9279	9284	32
58	9284	9289	9294	9298	9303	9308	9312	9317	9322	9326	9331	31
59	9331	9335	9340	9344	9349	9353	9358	9362	9367	9371	$\bar{1}$.9375	30°
60°	$\bar{1}$.9375	9380	9384	9388	9393	9397	9401	9406	9410	9414	9418	29
61	9418	9422	9427	9431	9435	9439	9443	9447	9451	9455	9459	28
62	9459	9463	9467	9471	9475	9479	9483	9487	9491	9495	9499	27
63	9499	9503	9506	9510	9514	9518	9522	9525	9529	9533	9537	26
64	9537	9540	9544	9548	9551	9555	9558	9562	9566	9569	9573	25
65	9573	9576	9580	9583	9587	9590	9594	9597	9601	9604	9607	24
66	9607	9611	9614	9617	9621	9624	9627	9631	9634	9637	9640	23
67	9640	9643	9647	9650	9653	9656	9659	9662	9666	9669	9672	22
68	9672	9675	9678	9681	9684	9687	9690	9693	9696	9699	9702	21
69	9702	9704	9707	9710	9713	9716	9719	9722	9724	9727	$\bar{1}$.9730	20°
70°	$\bar{1}$.9730	9733	9735	9738	9741	9743	9746	9749	9751	9754	9757	19
71	9757	9759	9762	9764	9767	9770	9772	9775	9777	9780	9782	18
72	9782	9785	9787	9789	9792	9794	9797	9799	9801	9804	9806	17
73	9806	9808	9811	9813	9815	9817	9820	9822	9824	9826	9828	16
74	9828	9831	9833	9835	9837	9839	9841	9843	9845	9847	9849	15
75	9849	9851	9853	9855	9857	9859	9861	9863	9865	9867	9869	14
76	9869	9871	9873	9875	9876	9878	9880	9882	9884	9885	9887	13
77	9887	9889	9891	9892	9894	9896	9897	9899	9901	9902	9904	12
78	9904	9906	9907	9909	9910	9912	9913	9915	9916	9918	9919	11
79	9919	9921	9922	9924	9925	9927	9928	9929	9931	9932	$\bar{1}$.9934	10°
80°	$\bar{1}$.9934	9935	9936	9937	9939	9940	9941	9943	9944	9945	9946	9
81	9946	9947	9949	9950	9951	9952	9953	9954	9955	9956	9958	8
82	9958	9959	9960	9961	9962	9963	9964	9965	9966	9967	9968	7
83	9968	9968	9969	9970	9971	9972	9973	9974	9975	9975	9976	6
84	9976	9977	9978	9978	9979	9980	9981	9981	9982	9983	9983	5
85	9983	9984	9985	9985	9986	9987	9987	9988	9988	9989	9989	4
86	9989	9990	9990	9991	9991	9992	9992	9993	9993	9994	9994	3
87	9994	9994	9995	9995	9996	9996	9996	9996	9997	9997	9997	2
88	9997	9998	9998	9998	9998	9999	9999	9999	9999	9999	$\bar{1}$.9999	1
89	$\bar{1}$.9999	9999	0000	0000	0000	0000	0000	0000	0000	0000	0.0000	0°
90°	0.0000											

Log Tan **Log Cotan**

	0	.1	.2	.3	.4	.5	.6	.7	.8	.9		
		.9	.8	.7	.6	.5	.4	.3	.2	.1	.0	
											−∞	90°
0°	−∞	$\bar{3}$.2419	5429	7190	8439	9409 ∫	0200	0870	1450	1962	$\bar{2}$.2419	89
1	$\bar{2}$.2419	2833	3211	3559	3881	4181	4461	4725	4973	5208	5431	88
2	5431	5643	5845	6038	6223	6401	6571	6736	6894	7046	7194	87
3	7194	7337	7475	7609	7739	7865	7988	8107	8223	8336	8446	86
4	8446	8554	8659	8762	8862	8960	9056	9150	9241	9331	$\bar{2}$.9420	85
5	$\bar{2}$.9420	9506	9591	9674	9756	9836	9915	9992 ∫	0068	0143	$\bar{1}$.0216	84
6	$\bar{1}$.0216	0289	0360	0430	0499	0567	0633	0699	0764	0828	0891	83
7	0891	0954	1015	1076	1135	1194	1252	1310	1367	1423	1478	82
8	1478	1533	1587	1640	1693	1745	1797	1848	1898	1948	1997	81
9	1997	2046	2094	2142	2189	2236	2282	2328	2374	2419	$\bar{1}$.2463	80°
10°	$\bar{1}$.2463	2507	2551	2594	2637	2680	2722	2764	2805	2846	2887	79
11	2887	2927	2967	3006	3046	3085	3123	3162	3200	3237	3275	78
12	3275	3312	3349	3385	3422	3458	3493	3529	3564	3599	3634	77
13	3634	3668	3702	3736	3770	3804	3837	3870	3903	3935	3968	76
14	3968	4000	4032	4064	4095	4127	4158	4189	4220	4250	4281	75
15	4281	4311	4341	4371	4400	4430	4459	4488	4517	4546	4575	74
16	4575	4603	4632	4660	4688	4716	4744	4771	4799	4826	4853	73
17	4853	4880	4907	4934	4961	4987	5014	5040	5066	5092	5118	72
18	5118	5143	5169	5195	5220	5245	5270	5295	5320	5345	5370	71
19	5370	5394	5419	5443	5467	5491	5516	5539	5563	5587	$\bar{1}$.5611	70°
20°	$\bar{1}$.5611	5634	5658	5681	5704	5727	5750	5773	5796	5819	5842	69
21	5842	5864	5887	5909	5932	5954	5976	5998	6020	6042	6064	68
22	6064	6086	6108	6129	6151	6172	6194	6215	6236	6257	6279	67
23	6279	6300	6321	6341	6362	6383	6404	6424	6445	6465	6486	66
24	6486	6506	6527	6547	6567	6587	6607	6627	6647	6667	6687	65
25	6687	6706	6726	6746	6765	6785	6804	6824	6843	6863	6882	64
26	6882	6901	6920	6939	6958	6977	6996	7015	7034	7053	7072	63
27	7072	7090	7109	7128	7146	7165	7183	7202	7220	7238	7257	62
28	7257	7275	7293	7311	7330	7348	7366	7384	7402	7420	7438	61
29	7438	7455	7473	7491	7509	7526	7544	7562	7579	7597	$\bar{1}$.7614	60°
30°	$\bar{1}$.7614	7632	7649	7667	7684	7701	7719	7736	7753	7771	7788	59
31	7788	7805	7822	7839	7856	7873	7890	7907	7924	7941	7958	58
32	7958	7975	7992	8008	8025	8042	8059	8075	8092	8109	8125	57
33	8125	8142	8158	8175	8191	8208	8224	8241	8257	8274	8290	56
34	8290	8306	8323	8339	8355	8371	8388	8404	8420	8436	8452	55
35	8452	8468	8484	8501	8517	8533	8549	8565	8581	8597	8613	54
36	8613	8629	8644	8660	8676	8692	8708	8724	8740	8755	8771	53
37	8771	8787	8803	8818	8834	8850	8865	8881	8897	8912	8928	52
38	8928	8944	8959	8975	8990	9006	9022	9037	9053	9068	9084	51
39	9084	9099	9115	9130	9146	9161	9176	9192	9207	9223	$\bar{1}$.9238	50°
40°	$\bar{1}$.9238	9254	9269	9284	9300	9315	9330	9346	9361	9376	9392	49
41	9392	9407	9422	9438	9453	9468	9483	9499	9514	9529	9544	48
42	9544	9560	9575	9590	9605	9621	9636	9651	9666	9681	9697	47
43	9697	9712	9727	9742	9757	9772	9788	9803	9818	9833	$\bar{1}$.9848	46
44	$\bar{1}$.9848	9864	9879	9894	9909	9924	9939	9955	9970	9985 ∫	0.0000	45°
45°	0.0000											

Log Tan											Log Cotan
		.9	.8	.7	.6	.5	.4	.3	.2	.1	.0
	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	
										0.0000	45°
45°	0.0000	0015	0030	0045	0061	0076	0091	0106	0121	0136	0152	44
46	0152	0167	0182	0197	0212	0228	0243	0258	0273	0288	0303	43
47	0303	0319	0334	0349	0364	0379	0395	0410	0425	0440	0456	42
48	0456	0471	0486	0501	0517	0532	0547	0562	0578	0593	0608	41
49	0608	0624	0639	0654	0670	0685	0700	0716	0731	0746	0.0762	40°
50°	0.0762	0777	0793	0808	0824	0839	0854	0870	0885	0901	0916	39
51	0916	0932	0947	0963	0978	0994	1010	1025	1041	1056	1072	38
52	1072	1088	1103	1119	1135	1150	1166	1182	1197	1213	1229	37
53	1229	1245	1260	1276	1292	1308	1324	1340	1356	1371	1387	36
54	1387	1403	1419	1435	1451	1467	1483	1499	1516	1532	1548	35
55	1548	1564	1580	1596	1612	1629	1645	1661	1677	1694	1710	34
56	1710	1726	1743	1759	1776	1792	1809	1825	1842	1858	1875	33
57	1875	1891	1908	1925	1941	1958	1975	1992	2008	2025	2042	32
58	2042	2059	2076	2093	2110	2127	2144	2161	2178	2195	2212	31
59	2212	2229	2247	2264	2281	2299	2316	2333	2351	2368	0.2386	30°
60°	0.2386	2403	2421	2438	2456	2474	2491	2509	2527	2545	2562	29
61	2562	2580	2598	2616	2634	2652	2670	2689	2707	2725	2743	28
62	2743	2762	2780	2798	2817	2835	2854	2872	2891	2910	2928	27
63	2928	2947	2966	2985	3004	3023	3042	3061	3080	3099	3118	26
64	3118	3137	3157	3176	3196	3215	3235	3254	3274	3294	3313	25
65	3313	3333	3353	3373	3393	3413	3433	3453	3473	3494	3514	24
66	3514	3535	3555	3576	3596	3617	3638	3659	3679	3700	3721	23
67	3721	3743	3764	3785	3806	3828	3849	3871	3892	3914	3936	22
68	3936	3958	3980	4002	4024	4046	4068	4091	4113	4136	4158	21
69	4158	4181	4204	4227	4250	4273	4296	4319	4342	4366	0.4389	20°
70°	0.4389	4413	4437	4461	4484	4509	4533	4557	4581	4606	4630	19
71	4630	4655	4680	4705	4730	4755	4780	4805	4831	4857	4882	18
72	4882	4908	4934	4960	4986	5013	5039	5066	5093	5120	5147	17
73	5147	5174	5201	5229	5256	5284	5312	5340	5368	5397	5425	16
74	5425	5454	5483	5512	5541	5570	5600	5629	5659	5689	5719	15
75	5719	5750	5780	5811	5842	5873	5905	5936	5968	6000	6032	14
76	6032	6065	6097	6130	6163	6196	6230	6264	6298	6332	6366	13
77	6366	6401	6436	6471	6507	6542	6578	6615	6651	6688	6725	12
78	6725	6763	6800	6838	6877	6915	6954	6994	7033	7073	7113	11
79	7113	7154	7195	7236	7278	7320	7363	7406	7449	7493	0.7537	10°
80°	0.7537	7581	7626	7672	7718	7764	7811	7858	7906	7954	8003	9
81	8003	8052	8102	8152	8203	8255	8307	8360	8413	8467	8522	8
82	8522	8577	8633	8690	8748	8806	8865	8924	8985	9046	9109	7
83	9109	9172	9236	9301	9367	9433	9501	9570	9640	9711	0.9784	6
84	0.9784	9857	9932	0008	0085	0164	0244	0326	0409	0494	1.0580	5
85	1.0580	0669	0759	0850	0944	1040	1138	1238	1341	1446	1554	4
86	1554	1664	1777	1893	2012	2135	2261	2391	2525	2663	2806	3
87	2806	2954	3106	3264	3429	3599	3777	3962	4155	4357	4569	2
88	4569	4792	5027	5275	5539	5819	6119	6441	6789	7167	1.7581	1
89	1.7581	8038	8550	9130	9800	2.0591	1561	2810	4571	7581	∞	0°
90°	∞											

From Minutes and Seconds into Decimal Parts of a Degree

′	°
0′	=0°.00 000
1′	.01 666..
2′	.03 333..
3′	.05
4′	.06 666..
5′	.08 333..
6′	.10
7′	.11 666..
8′	.13 333..
9′	.15
10′	0°.16 666..
1	.18 333..
2	.20
3	.21 666..
4	.23 333..
15′	.25
6	.26 666..
7	.28 333..
8	.30
9	.31 666..
20′	0°.33 333..
1	.35
2	.36 666..
3	.38 333..
4	.40
25′	.41 666..
6	.43 333..
7	.45
8	.46 666..
9	.48 333..
30′	0°.50
1	.51 666..
2	.53 333..
3	.55
4	.56 666..
35′	.58 333..
6	.60
7	.61 666..
8	.63 333..
9	.65
40′	0°.66 666..
1	.68 333..
2	.70
3	.71 666..
4	.73 333..
45′	.75
6	.76 666..
7	.78 333..
8	.80
9	.81 666..
50′	0°.83 333..
1	.85
2	.86 666..
3	.88 333..
4	.90
55′	.91 666..
6	.93 333..
7	.95
8	.96 666..
9	.98 333..
60′	1°.00

The dots (..) indicate that the last figure repeats indefinitely.

″	°
0″	=0°.00 000
1″	.00 027..
2″	.00 055..
3″	.00 083..
4″	.00 111..
5″	.00 138..
6″	.00 166..
7″	.00 194..
8″	.00 222..
9″	.00 25
10″	0°.00 277..
1	.00 305..
2	.00 333..
3	.00 361..
4	.00 388..
15″	.00 416..
6	.00 444..
7	.00 472..
8	.00 5
9	.00 527..
20″	0°.00 555..
1	.00 583..
2	.00 611..
3	.00 638..
4	.00 666..
25″	.00 694..
6	.00 722..
7	.00 75
8	.00 777..
9	.00 805..
30″	0°.00 833..
1	.00 861..
2	.00 888..
3	.00 916..
4	.00 944..
35″	.00 972..
6	.01
7	.01 027..
8	.01 055..
9	.01 083..
40″	0°.01 111..
1	.01 138..
2	.01 166..
3	.01 194..
4	.01 222..
45″	.01 25
6	.01 277..
7	.01 305..
8	.01 333..
9	.01 361..
50″	0°.01 388..
1	.01 416..
2	.01 444..
3	.01 472..
4	.01 5
55″	.01 527..
6	.01 555..
7	.01 583..
8	.01 611..
9	.01 638..
60″	0°.01 666..

From Decimal Parts of a Degree into Minutes and Seconds
(Exact Values)

0°.00	= 0′00″	0°.50	=30
1	0′36″	1	30′36″
2	1′12″	2	31′12″
3	1′48″	3	31′48″
4	2′24″	4	32′24″
0°.05	3′	0°.55	33′
6	3′36″	6	33′36″
7	4′12″	7	34′12″
8	4′48″	8	34′48″
9	5′24″	9	35′24″
0°.10	6′	0°.60	36′
1	6′36″	1	36′36″
2	7′12″	2	37′12″
3	7′48″	3	37′48″
4	8′24″	4	38′24″
0°.15	9′	0°.65	39′
6	9′36″	6	39′36″
7	10′12″	7	40′12″
8	10′48″	8	40′48″
9	11′24″	9	41′24″
0°.20	12′	0°.70	42′
1	12′36″	1	42′36″
2	13′12″	2	43′12″
3	13′48″	3	43′48″
4	14′24″	4	44′24″
0°.25	15′	0°.75	45′
6	15′36″	6	45′36″
7	16′12″	7	46′12″
8	16′48″	8	46′48″
9	17′24″	9	47′24″
0°.30	18′	0°.80	48′
1	18′36″	1	48′36″
2	19′12″	2	49′12″
3	19′48″	3	49′48″
4	20′24″	4	50′24″
0°.35	21′	0°.85	51′
6	21′36″	6	51′36″
7	22′12″	7	52′12″
8	22′48″	8	52′48″
9	23′24″	9	53′24″
0°.40	24′	0°.90	54′
1	24′36″	1	54′36″
2	25′12″	2	55′12″
3	25′48″	3	55′48″
4	26′24″	4	56′24″
0°.45	27′	0°.95	57′
6	27′36″	6	57′36″
7	28′12″	7	58′12″
8	28′48″	8	58′48″
9	29′24″	9	59′24″
0°.50	30′	1°.00	60′

0°.000	= 0″.0
1	3″.6
2	7″.2
3	10″.8
4	14″.4
0°.005	18″.
6	21″.6
7	25″.2
8	28″.8
9	32″.4
0°.010	36″.

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